Physics 239

Lecture 8 - Magnetorotational Instability (MRI)

→ Why MRI? → Mechanism for \( \nabla \)

→ MRE Physics

→ Simplified MRI / Heuristics.

→ Calculation. TB
Why MRI?

- Need mechanism to produce turbulence to exert shear stresses.
- Viscosity - due to fluctuations.
- $\textbf{T}_{\eta} = \sum r \frac{\partial \textbf{V}}{\partial r} \frac{\partial \eta}{\partial r}$
- $\eta \approx \eta_{\text{mix}}$
- $\eta_{\text{mix}}$

Hydro-turbulence:

$$\left< \textbf{u} \cdot \textbf{u} \right> \rightarrow \textbf{V}$$

Turbulent Reynolds stress.

MHD:

$$\left< \textbf{B} \cdot \textbf{B} \right> - \left< \textbf{B} \cdot \textbf{B} \right> = \textbf{V}$$

MRI = Algebraically weighted Reynolds + Maxwell stress.

Is there an instability which taps free energy in $\partial S/\partial r$?

Such as source/driver of turbulent transport = viscosity mechanism.
MRI Physics

Recall cartoon derivation from Lynden-Bell 2 particle argument.

Key point: 2 particles conserve total \( E \) but can exchange.

Then differential notation:

\[ \text{critical perturbation \textit{then-ahead}} \]

1. \( \rightarrow \) spring stretches
2. outer behind

Spring coupler between particles
but the particles are "donkeys" → do the opposite of the judge

\[ \frac{\beta}{2} \]

losses angular momentum
drops to inner orbit with higher \( \beta \)

\[ \frac{\gamma}{2} \]

gains angular momentum
moves to outer orbit with lower \( \gamma \)

$\Rightarrow$ moves further ahead

$\Rightarrow$ drops further behind

$\Rightarrow$ stretching the spring results in further stretching of spring
Δ perturbation self-reinforces.

Δ instability

Note:

"spring facilitates" instability via interaction, but is not the free energy source.

ΔL is what drives instability.

D' < 0 matters.
- Observe contrast between:

$$\mathbf{J} = 2 R \frac{d}{dr} (r^2 \mathbf{L})$$

$$< 0$$

$$> 0$$

- L-B-P arguments apply $\Rightarrow$
  angular momentum transported outward.

- Noting: Spring <em>η<sub>p</sub></em> $\Rightarrow$ Magnetic Tension.

  attachment $\Rightarrow$ Freezing-in Law.

- MRI mechanism is simple, robust
  linear instability mechanism leading
to turbulence and angular momentum
  transport in keplerian disks

  which are counter $\Rightarrow$ hot

- Caveat Emptor: Photo-planetary disks.
MRT: Simplified Model

Consider ideal MRT:

\[ \frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla P + \nabla \times (\mu \nabla \times u) + \frac{B \cdot \nabla B}{\mu_0} \]

\[ \frac{\partial B}{\partial t} + \nabla \times (u \times B) = 0 \]

and disk, threaded by vertical field

\[ \text{Gravity} \]

\[ \text{2 cases:} \]
- Vertical field
  \[ B = B_0 \frac{z}{\mu_0} \]
- Magnetic
  \[ B = B_0 \sqrt{v} \]

Consider:

Perturb fluid element attached to B0.
then, on plane disk:

Consider as mechanical problem, i.e.

Particle motion on a gravitational potential and acted on by magnetic forces (tension)

\[ L = \frac{1}{2} \left( r^2 + r^2 \dot{\phi}^2 \right) - \Phi = U_B \]

\[ \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0 \]

\[ \ddot{r} - r \dot{\phi}^2 = -\frac{d\Phi}{dr} - \frac{dU_B}{dr} \]

Generalized force.
\[ \ddot{r} - r \dot{\phi}^2 = -\frac{\partial \Phi}{\partial r} + f_r \]

and

\[ \frac{1}{r} \frac{d}{dr} \left( r^2 \dot{\phi} \right) = -\frac{2\Phi}{r} \]

\[ r^{\ddot{\phi}} + 2r \dot{r} \dot{\phi} = -\frac{\partial \Phi}{\partial \phi} \]

\[ r^{\ddot{\phi}} + 2r \dot{r} \dot{\phi} = \frac{\gamma}{r} \]

Then, slab made into shearing sheet

\[ r = \sqrt{\alpha + x} \]
\[ \phi = \gamma t + \frac{y}{\sqrt{\alpha}} \]
\[ y = \alpha (\phi - 2t) \]

Then,
\[ x'' = -(\rho + x) \left( \frac{\rho + x}{b^2} \right)^2 = -\frac{GM}{(\rho + x)^2} \]

\[ x' = -\rho \frac{\rho^2}{b^2} - \rho^2 \frac{\rho^2}{b^3} \frac{\rho}{\rho} + \text{h.o.t.} \]

\[ = -\frac{GM}{b^2} + \frac{GM}{b^2} \left( \frac{2\rho}{b} \right) + f_x \]

\[ \Rightarrow \text{repulsive} \quad \frac{\text{backward}}{\text{h.o.}} \]

\[ x'' - 2\rho \frac{\rho^2}{b^2} = \frac{GM}{b^2} \frac{2\rho}{b} + f_x \]

\[ = \rho \frac{\rho^2}{b^2} \frac{2\rho}{b} + f_x \]

\[ = -x \frac{d\rho^2}{d\rho} + f_x \]

\[ \Rightarrow \nabla^2 \rho \leq 0. \]

\[ x'' - 2\rho \frac{\rho^2}{b^2} = -x \frac{d\rho^2}{d\rho} + f_x \]
\[(r_0 + x) \left( \frac{\partial^2}{\partial r^2} \right) + 2 x \left( \frac{\partial}{\partial x} \right) = f_y \]

\[y'' + 2x y' = f_y\]

**Finally:**

\[\dot{x} - 2 \omega \dot{y} = -x \frac{\partial L^2}{\partial \dot{x}} + f_x\]  
Coriolis

\[y'' + 2 \omega \dot{x} = f_y\]

**What are \(f_x, f_y\)?**

- Forces result from "plucked" magnetic fields i.e. magnetic tension.

- Recall Alfven wave:

\[\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v} \times \mathbf{B} + \frac{B_0}{4\pi \rho_0} \mathbf{B}\]
\[ B = B_0 \frac{z^2}{r^2} \]

\[ \nabla^2 \psi = \left( \frac{B_0}{4\pi} \right)^2 \nabla^2 \psi \]

\[ \psi = \psi_0 \frac{z}{r} \]

\[ \text{Ld fluid element displacement} \]

\[ \psi_0 \frac{z}{r} = \eta A_z \frac{\partial^2}{\partial z^2} \frac{z}{r} \Delta \xi \]

\[ \Delta \xi = \xi e^{i(\xi z - \omega t)} \]

\[ \xi = \xi_0 \]

\[ \text{force per unit mass} \]

\[ \xi = (x, y) \]

So

\[ F_x = -\eta^2 \nu A_z^2 x \]

\[ F_y = -\eta^2 \nu A_z^2 y. \]
\[ \dot{\ddot{x}} - 2\Omega \dot{y} = -x \frac{1}{\text{d}e^{\text{r}}} \left( \frac{1}{\text{d}e^{\text{r}}} \right) - (kVA)^2 x \]

\[ \ddot{y} + 2\Omega \dot{x} = -(kVA)^2 y \]

\[ x = (x_0) e^{-\omega t} \]

\[ y = (y_0) e^{-\omega t} \]

\[ -\omega^2 x_0 + 2i\omega \Omega y_0 = -x_0 \frac{1}{\text{d}e^{\text{r}}} - (kVA)^2 x_0 \]

\[ -\omega^2 y_0 - 2i\omega \Omega x_0 = -(kVA)^2 y_0 \]

\[ \text{N.b.} \text{ 4 transient rotation:} \]

\[ -\omega^2 x_0 = -(kVA)^2 x_0 \]

\[ -\omega^2 y_0 = -(kVA)^2 y_0 \]

\[ \Rightarrow \text{Shear Alkam wave!} \]
Paleont B-Field:

\[
\begin{vmatrix}
(- \omega^2 + \frac{d \Omega^2}{d \ln r}) x_0, & 2i \omega \Omega \\
-2i \omega \omega, & -\omega^2
\end{vmatrix} = 0
\]

\[
(-\omega^2)(-\omega^2 + \frac{d \Omega^2}{d \ln r}) - 4 \omega^2 \Omega^2 = 0
\]

\[-\omega^2 + \frac{d \Omega^2}{d \ln r} + 4 \Omega^2 = 0\]

\[\omega^2 = 4 \Omega^2 + \frac{d \Omega^2}{d \ln r}\]

\[= 4 \Omega^2 + \frac{d \Omega^2}{d \ln r} \]

\[= 4 \Omega^2 - 3 \Omega^2 \]

\[> 0\]

\(r^2 = \frac{\Omega^2}{r^2}\)

\(1\) B-Field

facilitates energy transfer to stable rotational oscillation, perhaps coupling...
So, determinant of full system:

\[
\begin{align*}
(\omega^2)^2 - \omega^2 \left[ \left( \frac{d}{dr} \right)^2 + 4 + \frac{2}{\text{den}} \right] \\
+ (\text{kVA})^2 \left[ (\text{kVA})^2 + \frac{\text{den}}{\text{den}} \right] = 0
\end{align*}
\]

\[X^2 + BX + C = 0 \quad X = \omega^2\]

\[\omega^2 = -\frac{B}{2} \pm \frac{1}{2} \left( \frac{B^2 - 4C}{2} \right)^{1/2}\]

\[\omega^2 > 0 \rightarrow \text{stable} \quad \rightarrow \text{oscillatory wave solution}\]

\[\omega^2 < 0 \rightarrow \text{unstable} \quad \rightarrow \text{growing perturbations}\]

\[B < 0 \quad \text{Need} \quad \frac{C}{\text{den}} < 0 \Rightarrow \text{so} \quad \frac{-B}{2} = \frac{1}{2} \left( \frac{B^2 - 4C}{4} \right)^{1/2}\]

\[\Rightarrow \quad \text{root negative} \]
For tokamak:

\[ (k_0 V_a)^2 - (V_a / a)^2 \]

\[ dL^2 / dh = d(V_a / R)^2 / dh \]

\[ V_a >> V_T. \]

Field is facilitated not driven

\[ \Rightarrow \text{Field must be "weak"} \]

For Keplerian:

\[ (k^2 V_a)^2 - 3 R^2 < 0 \]

\[ \Rightarrow \text{unstable for } k < k_{crit} \]

\[ k_{crit} V_a = \sqrt{3} R \]

\[ \Rightarrow \text{smallest scale of instability is} \]

\[ \frac{\lambda_{min}}{2} = 2\pi / k_{crit} \]

\[ = 2\pi / \sqrt{3} R / V_a \]

\[ = (\frac{2\pi}{\sqrt{3}}) \frac{V_a}{R} \]
If \( x_{min} \geq 2h \Rightarrow \text{mode does not fit in dish.} \quad \text{- analysis}

\[
B = \left( \frac{2\pi}{13} \right) \frac{v_A}{r} < 2h = 2 \frac{C_s}{r}
\]

\[
\Rightarrow (\frac{C_s}{v_A})^2 > \# \sim 2 \frac{4\pi^2}{3}
\]

Need

\[
B > 2 \frac{4\pi^2}{3}
\]

high B system!

\[
B = \frac{C_s^3}{vA^2}
\]

- \text{Magnetic Pressure}

M.B. \quad - B \sim 1 \text{ (strongly magnetic) - equipartition} \Rightarrow \text{stable}

- MRI in weak field instability

- MRI is strong!

\[
(w^2)^2 - w^3 B + C = 0
\]
\[ 2(\omega^2) \frac{d(\omega^2)}{d(\kappa \nu A)} - B \frac{d(\omega^2)}{d(\kappa \nu A)} - \omega^2 \frac{dB}{d(\kappa \nu A)} + \frac{dC}{d(\kappa \nu A)} = 0 \]

\[ \frac{d(\omega^2)}{d(\kappa \nu A)} = \frac{\omega^2 \frac{dB}{d(\kappa \nu A)} - \frac{dC}{d(\kappa \nu A)}}{2\omega^2 - B} = 0 \]

Crank \Rightarrow \]

\[ (\kappa \nu A)_{\text{max}} = \sqrt{\frac{4}{\omega^2}} \]

\[ \kappa \nu A \text{ for max growth} \]

\[ \delta_{\text{max}} = \frac{3}{4} \pi \]

\[ \mathcal{C} = \omega \]

\[ \Rightarrow \text{e-folding or rotation period} \]

\[ \Rightarrow \text{strong} \]

MRI is robust against instability.
Instability for:

\[(k \nu a)^2 + \frac{d \Omega^2}{d \ln r} < 0\]

Instability possible for \(\frac{d \Omega^2}{d \ln r} < 0\)

- Keplerian \(\checkmark\) possible
- Contrast to Rayleigh:
  \[\Phi = \frac{2 \Omega a^2 (k^2 \Omega)}{n} < 0\]

\(\Rightarrow\) differential rotation drive

Also observe:

- not relevant to strongly magnetized system.

We consider \((k \nu a)^2 + \frac{d \Omega^2}{d \ln r} < 0\) Condition.
Remaining Questions:

1. Linear Theory:
   - Effects lower temp weak convection?
   - Resistivity
   - Ambi-polar diffusion → down \( B \)

2. Angular momentum transport and profile back-react\( \text{ion} \)

\[
C E \quad \nu = \alpha C S H
\]

Sakura - Sunyaev, \( \alpha \) \( p \)h

\[
x = \frac{\langle \dot{V} \rangle \dot{a}^2}{C S^2} - \frac{\langle Br \rangle^2}{4\pi \beta C S^2} \]

→ calculate

N.B. Physics = magnetic stress dominant.
- Can relate to displacement $E$.

$\frac{d^2}{dt^2}$ is done.

- Expect instability relaxes $L$.

- Departure Keplerian profile?

  How much? $\rightarrow$ Gravity

$\rightarrow$ Interesting C$^2$L dynamics problem.

- Turbulence

  - Alfvenic + strong differential rotation.

  - Not simple $\nabla \cdot T$.

- Some controversy about accuracy of $\omega$, $\omega$.

  $\Rightarrow$ What is NL state like?