At the top of your submission, list all the sources you consulted, or write "Sources consulted: none" if you did not consult any sources.

1. Let \( R \subseteq \mathbb{C}[x] \) be the subring of polynomials \( P \) such that the coefficient of \( x \) in \( P \) is zero.
   
   (a) (1 point) Give an embedding of \( \text{Spec} R \) into \( \mathbb{A}^2 \), and show that the image has a cusp.
   
   (b) (1 point) Find a smooth curve \( \text{Spec} S \) with a map \( \text{Spec} S \to \text{Spec} R \) which is an isomorphism on topological spaces. Observe that this means that the composition \( \text{Spec} S \to \text{Spec} R \to \mathbb{A}^2 \) is a closed embedding of topological spaces but not a closed embedding of algebraic varieties.

2. (2 points) Let \( R \) be a finite type \( \mathbb{C} \)-algebra that is integral (i.e., has no zero-divisors.) Let \( S \) be a multiplicative system in \( R \). Show that the localization \( R_S \) is a finite type \( \mathbb{C} \)-algebra if and only if it is isomorphic to the localization \( R_f \) at a single nonzero element \( f \). (Recall that \( R_f \) is the localization of \( R \) at the multiplicative system \( \{1, f, f^2, \cdots \} \).

3. Our definition of \( \text{Spec} R \) as a topological space still makes sense for rings \( R \) which are not finite type \( \mathbb{C} \)-algebras. We will not worry too much about such algebras in this class, but let us briefly discuss the case of \( \mathbb{R} \)-algebras.
   
   (a) (1 point) Classify the maximal ideals of \( \mathbb{R}[x] \), and describe the map
   
   \[ \text{Spec}(\mathbb{C}[x]) \to \text{Spec}(\mathbb{R}[x]). \]

   (b) (1 point) Classify the maximal ideals of \( \mathbb{R}[x,y]/(x^2 + y^2 + 1) \), and describe the map
   
   \[ \text{Spec}(\mathbb{C}[x,y]/(x^2 + y^2 + 1)) \to \text{Spec}(\mathbb{R}[x,y]/(x^2 + y^2 + 1)). \]

   Note that the vanishing locus of \( x^2 + y^2 + 1 = 0 \) in \( \mathbb{R}^2 \) is empty, and yet we can still study the algebraic geometry of this ring.
4. Let $S$ be a subset of $\mathbb{Z}^n$ containing 0 and closed under addition (in other words, a sub-semigroup of $\mathbb{Z}^n$). We can define a ring $\mathbb{C}[S]$ whose elements are formal linear combinations $\sum a_i t^s_i$ with the $s_i \in S$, with multiplication determined by the rule $t^s_i \cdot t^{s_j} = t^{s_i + s_j}$. An affine toric variety is the spectrum of a ring $\mathbb{C}[S]$. Toric varieties give a large family of easy examples of varieties.

(a) (1 point) Show that every inclusion $S \subseteq S'$ gives a map of toric varieties $\text{Spec} \mathbb{C}[S'] \to \text{Spec} \mathbb{C}[S]$.

(b) (1 point) Show that any toric variety has an open subset which is isomorphic to a torus (i.e., the spectrum of an algebra $\mathbb{C}[x_i, x_i^{-1}]$).

This is why these varieties are called toric.

5. (2 points) Recall in class that we mentioned that $X = \mathbb{A}^2 - \{(0,0)\}$ is not an affine variety. More precisely, we claim that there is no affine variety $Y$ with a map $\pi : Y \to \mathbb{A}^2$ and two open subvarieties $U$ and $V$ satisfying the following properties:

- $Y$ is the union of $U$ and $V$
- $\pi$ induces an isomorphism of varieties between $U$ (respectively, $V$) and the complement of the $x$-axis (respectively, the $y$-axis) in $\mathbb{A}^2$
- $\pi$ induces an isomorphism of varieties between the intersection $U \cap V$ and the locus where $xy$ does not vanish in $\mathbb{A}^2$.

Prove this. (Hint: One way of doing this is to think about maps from such a variety $Y$ to $\mathbb{A}^1$.)

6. (1 point) Look up the definition of a sheaf. Use google to find as many motivations as you can for why you would define such an object. Elaborate on the one you find most convincing.