# Math 108 - Geometric Combinatorics

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## Polytopes

1/23/23

A convex polytope $P \subseteq \mathbb{R}^d$ is the closure of a relatively bounded region of a hyperplane arrangement. Its dimension dim(P), is the rank of the subarrangement made of the hyperplane bordering P.

A hyperplane H is a supporting hyperplane if both
- P is contained in one of the two closed half-spaces created by H.
- $P \cap H \neq \emptyset$.

**Definition**

A convex polytope is the convex hull of finitely many points in $\mathbb{R}^d$.

smallest convex set containing them.

Given $x_1, \dots, x_n \in \mathbb{R}^d$, this is

$$P = \text{conv}\{x_1, \dots, x_n\} = \{\lambda_1 x_1 + \dots + \lambda_n x_n \mid \lambda_i \geq 0 \ \forall i \in [n], \lambda_1 + \dots + \lambda_n = 0\}$$

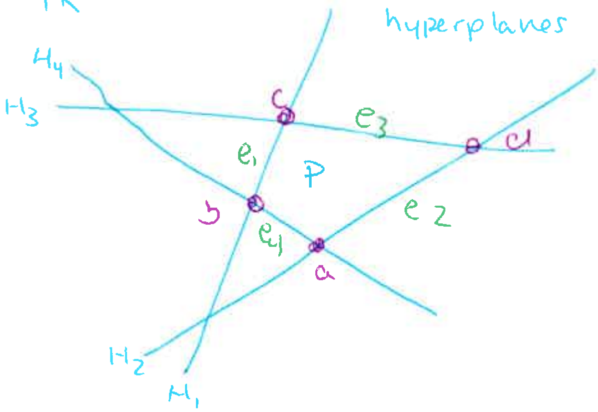
The equivalence of the definitions follows from a theorem (whose proof can be found in [CCD, Appendix A]).

**Definition**

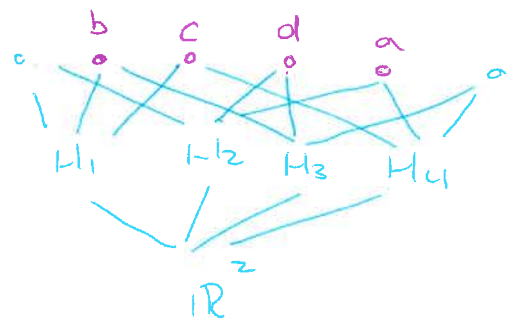
The faces of a polytope P are the intersection of hyperplanes intersected with P.

## Example

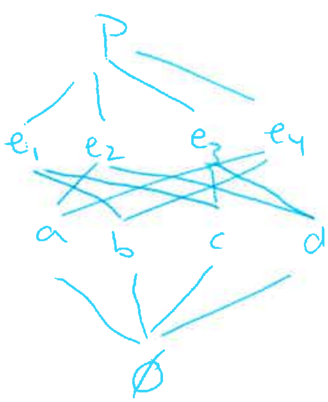
$\mathbb{R}^2$ $\mathcal{A}$: arrangement of hyperplanes




Intersection poset of $\mathcal{A}$




Poset of faces of P
(order is inclusion)




Rank $= \dim(F)+1$ for faces F

Faces are polytopes themselves

## Definition

Let P be a convex polytope.
The face lattice of a convex polytope is the poset of faces, partially ordered by inclusion.

## Theorem

For every polytope P, the face lattice $L(P)$ is a ranked poset of height $\dim(P)$, and with $\operatorname{rank}(F) = \dim(F)+1$ for every face.

## Definition

- Faces of dimension 0 are called vertices
- Faces of dimension 1 are edges
- Faces of dimension $d-1$ of a $d$-polytope are facets.
- A face is proper if it is not the polytope itself.
- Given a $d$-dimensional polytope, the f-vector of $P$ is the tuple $(f_0, f_1, \dots, f_{d-1})$, where $f_i$ is the number of faces of dimension $i$.

Counting question: What tuples of length $d$ can be the f-vector for $d$-dimensional polytopes?

## Dimension 1

- A polytope is an interval:


f-vector: $(2)$.

## Dimension 2

- A polytope is a polygon.
- It has as many vertices as edges: $(n,n)$, $n \geq 3$, is the set of admissible f-vectors.

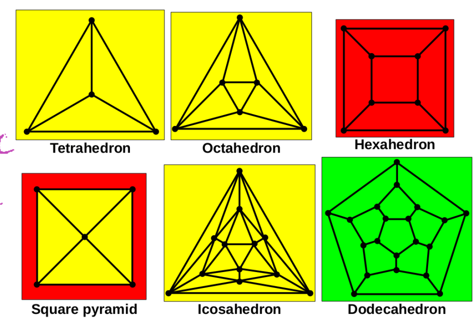
# Dimension 3.

Your observations:

Polytopes we know:

| P | f(P) |
|---|---|
| Cube | (8, 12, 6) |
| Tetrahedron | (4, 6, 4) |
| Octahedron | (6, 12, 8) |
| Dodecahedron | (20, 30, 12) |
| Icosahedron | (12, 30, 20) |
| n-sided pyramid | (n+1, 2n, n+1) |

Platonic Solids.




Observation (for $d=3$): $f_0 - f_1 + f_2 = 2$

Observation (in general): $f_0 - f_1 + f_2 - \dots + (-1)^{d-1} f_{d-1} = 1-(-1)^d$

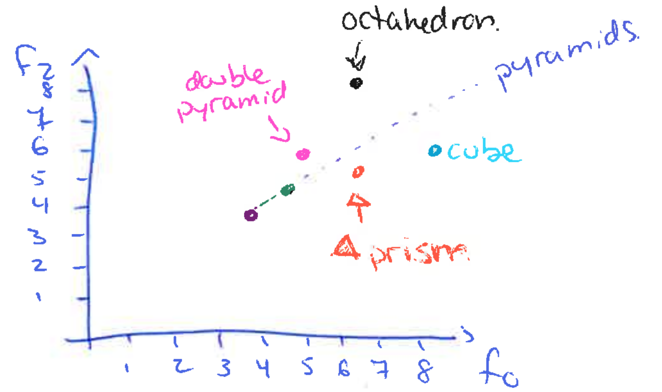
## Theorem (Euler characteristic)

The f-vector of a convex polytope satisfies $\sum_{k=0}^{d-1} (-1)^k f_k = 1-(-1)^d$, where $d$ is the dimension of the polytope.

## Corollary

The f-vector of a 3-dimensional polytope depends only on $f_0$ and $f_2$.

Can all $f_0$ and $f_2$ appear?

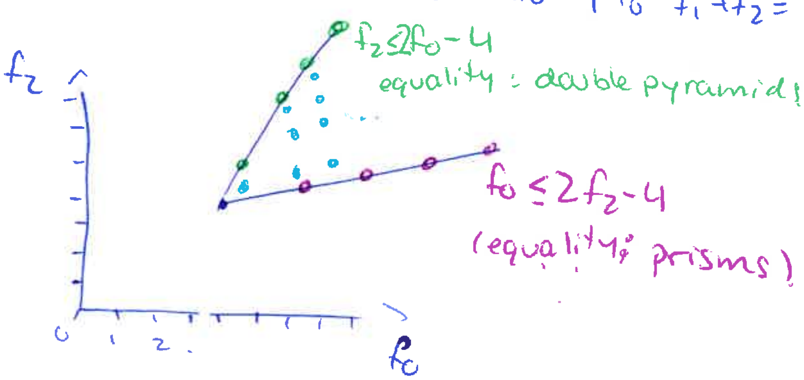



What other integer points can represent f-vectors?

## Lemma (Steinitz, 1906)

The set of all f-vectors of polytopes of dimension 3 is given by

$$\{(f_0, f_1, f_2) \in \mathbb{N}^3 \mid f_0 - f_1 + f_2 = 2,\ f_2 \le 2f_0 - 4,\ f_0 \le 2f_2 - 4\}.$$




## Lemma

For each polytope in dimension 3, there exists a <u>dual polytope</u> (taken by swapping vertices and faces) with f-vector $(f_2, f_1, f_0)$.

## Proof of Steinitz lemma

1. The first equation follows from Euler's formula.
2. We prove $f_2 \le 2f_0 - 4$.

Since each facet has at least 3 edges, and each edge belongs to two facets,

$$2f_1 = \sum_{f \in \text{facets}} \underbrace{\mathrm{len}(f)}_{\#\text{ edges}} \geq 3f_2$$

Using Euler's formula: $f_1 = f_0 + f_2 - 2$, and

$3f_2 \leq 2(f_0 + f_2 - 2) = 2f_0 + 2f_2 - 4$, which gives $f_2 \leq 2f_0 - 4$.

3. Using the dual polytope, $f_0 \leq 2f_2 - 4$.

4. We still need to prove that every tuple satisfying these conditions is the f-vector of a polytope.

Let $f = (f_0,\ f_0 + f_2 - 2,\ f_0 + k)$, $0 \leq k \leq f_0 - 4$. (so $f_0 - k \geq 4$).

Consider the pyramid with a base with $f_0 - k - 1$ sides. It has f-vector $(f_0 - k,\ 2(f_0 - k - 2),\ f_0 - k)$, and all $f_0 - k$ facets are triangles. By replacing any facet by a tetrahedron, we replace one facet by three facets, adding two faces, and 1 vertex.

We do it $k$ times (maybe in several iterations of the process), and get f-vector $(f_0 - k + k,\ f_0 + f_2 - 2,\ f_0 - k + 2k) = (f_0,\ 2f_0 + k - 2,\ f_0 + k)$, as desired.

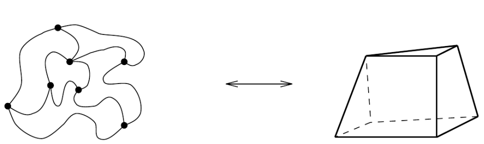
The case where $f_2 < f_0$ is obtained from the dual polytope.

One can do better, and actually characterize all graphs of polytopes:

**Theorem** (Steinitz, 1922)

There is a bijection: $\{$3-connected planar graphs$\}$ $\longleftrightarrow$ $\{$combinatorial description of polytopes of dimension 3$\}$

3-connected: for any two vertices $u, v$, there exists at least 3 disjoint paths from $u$ to $v$.



## Dimension 4

Can we describe the "cone" of f-vectors, similarly to Steinitz's lemma?

Theorem (Grünbaum, Barnette, Reay; 1973-2003)

No. It has concavities, and even holes.

## Other counting questions

Conjecture (disproved)

The f-vectors of convex polytopes are unimodal, e.g.

$$f_0 \le f_1 \le f_2 \le \dots \le f_\ell \ge \dots \ge f_{d-2} \ge f_{d-1}$$

Conjecture (Björner, 1981, open)

The f-vectors of convex polytopes increase for the first quarter, and decreases for the last.

$$f_0 \le f_1 \le \dots \le f_{\lceil \frac{d-1}{4} \rceil} \text{ and } f_{\lfloor \frac{3(d-1)}{4} \rfloor} \ge \dots \ge f_{d-1}$$

Conjecture (Bárány, open).

For any polytope, $f_k \ge \min(f_0, f_{d-1})$ for all $0 \le k \le d-1$.

References: [Zie] lectures 1,2,4.

[CCD] chapter 2