Announcements

MATH 302

Convex Sets
1. Convex Set
2. Convex Hull
3. Convex Functions

Theorem 1 (Convexity): For a set $C$ to be convex, if $a, b \in C$, then $ta + (1-t)b \in C$ for all $t \in [0, 1]$.

Theorem 2 (Convex Hull): The convex hull of a set $A$ is the smallest convex set containing $A$.

Example: The convex hull of the set of points $A = \{ (1, 0), (0, 1), (1, 1) \}$ is the triangle formed by these points.

Pigeonhole Principle

If $n > m$, then there exists at least one box with no balls.

Proof: Consider the $n$ boxes and the $n$ balls. If each box contains at most $m$ balls, then the total number of balls is at most $nm$. Since there are $n$ boxes, this means that at least one box must contain at least one more ball, violating the assumption.

Theorem 3 (Zermelo's Theorem): If $A$ is a non-empty set, then there exists a function $f: A \rightarrow A$ such that $f(x) \neq x$ for all $x \in A$.

Proof: Consider the set $B = \{ x \in A \mid f(x) = x \}$. By the Axiom of Choice, we can choose an element $a \in A$ such that $f(a) \neq a$. Then let $f(a) = b$. Since $b \neq a$, we have $f(b) \neq b$, and we can continue this process indefinitely.

Theorem 4 (Helly's Theorem): If $F$ is a finite family of convex sets in $\mathbb{R}^n$, then $F$ has the finite intersection property if and only if some finite subfamily of $F$ has the finite intersection property.

Proof: Assume $F$ has the finite intersection property and let $I$ be a finite subfamily of $F$. Then $\bigcap I \neq \emptyset$, and we can proceed by induction on the size of $I$.

Theorem 5 (Kuratowski's Theorem): If $A$ is a finite set in $\mathbb{R}^n$, then $A$ has a convex hull.

Proof: Consider the set $B = \{ x \in A \mid x \text{ is a vertex of } A \}$. By the Axiom of Choice, we can choose an element $a \in A$ such that $a \notin B$. Then let $B' = B \cup \{ a \}$. Since $B'$ is finite, it has a convex hull $C$. Then $A$ is contained in $C$, and we are done.