Recitation 5

Thursday September 22, 2022

1 Recap

1.1 Vector Space

A vector space $V$ is a set that is closed under vector addition and scalar multiplication. These two operations must satisfy a set of axioms. A basic example is the real vector space $\mathbb{R}^n$, where

- every vector is represented by a list of $n$ real numbers
- scalars are real numbers
- addition is defined element-wise
- scalar multiplication is multiplication on each term separately

For a general vector space, the scalars are elements of a field $F$ (e.g., the complex numbers), in which case $V$ is called a vector space over $F$. If $W \subseteq V$ is also a vector space with respect to the operations in $V$, then $W$ is called a subspace of $V$.

**Key Fact.** If $S_1$ and $S_2$ both are vector space, then $S = S_1 \cap S_2$ is a subspace.

1.2 Column Space

The column space of an $m \times n$ matrix $A$, denoted $C(A)$, is the set of all linear combinations of columns of $A$, or the span of $A$.

- $C(A) = \{Ax \mid x \in \mathbb{R}^n\}$.
- $Ax = b$ has a solution iff $b \in C(A)$.
- If $m = n$, then $A$ is invertible iff $C(A) = \mathbb{R}^n$.

1.3 Null Space

The null space of $A$, denoted $N(A)$, is the set of vectors $x$ such that $Ax = 0$.

- $N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$.
- If $B$ is a square and invertible matrix, then $N(A) = N(BA)$.
- If $A$ is $n \times n$, then $C(A) = \mathbb{R}^n \iff N(A) = \{0\} \iff A$ is invertible.

1https://en.wikipedia.org/wiki/Vector_space

2 Exercises

1. Is $V$ a real vector space?
   (a) $V$ is the set of all $n$-dimensional vectors with positive entries, with usual vector operations.
   (b) $V$ is the set of all $n$-dimensional vectors whose elements sum to 0, with usual vector operations.
   (c) $V$ is the set of all $n \times n$ diagonal matrices, with usual matrix operations.
   (d) $V$ is the set of all polynomials with degree up to $d$, with usual polynomial operations.
   (e) $V$ is the set of all constant functions, i.e. $f(x) = c$ for some constant $c \in \mathbb{R}$, with usual real number operations.
   (f) $V$ is the set of all single-variable polynomial whose value at 0 is 1, i.e. polynomial $P$ with $P(0) = 1$, with usual polynomial operations.

2. True or False
   (a) Define the row space of matrix $A$ as the span of the row vectors of $A$. If $A$ is a square matrix, then the row space of $A$ equals the column space.
   (b) The row space of $A$ is equal to the column space of $A^T$.
   (c) If the row space of $A$ equals the column space, then $A^T = A$.
   (d) A 4 by 4 permutation matrix has column space equal to $\mathbb{R}^4$.
   (e) Let $v \in N(A)$. If $x$ is a solution to equation $Ax = b$, so is $x + v$.

3. $A$ is a 3 by 3 matrix.
   (a) $x = [1 \ 2 \ 3]^T$ is in the null space of some matrix $A$. What is $A(2x)$?
   (b) $y = [1 \ 2 \ 4]^T$ is also in the null space of $A$. What is $A(2x + y)$?
   (c) $Az = [1 \ 2 \ 5]^T$. What is $A(2x + y + z)$

4. $Ax = b$, $A$ is a 3 by 3 matrix, $x$ and $b$ are 3-dimensional vectors.
   (a) When $b = [1 \ 2 \ 3]^T$ we have infinite solutions. Is $b$ in the column space of $A$?
   (b) Assume (a) is still true. Is $A$ invertible?
   (c) Now assume $A$ is invertible, do we have a solution when $b = 0$? How many?

5.
   
   \[
   A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}
   \]

   (a) Find a nonzero vector in the null space of $A$, if any exists.
   (b) Is $A$ invertible?
   (c) Find a vector in the column space of $A$ that is not equal to either column.
   (d) Find a vector in the row space of $A$ that is not equal to either row.