Yangians, their representations, and applications

Lecture 1

Course overview

Recollection on representations of GL(n)

Schur-Weyl duality

1. Main idea: Yangian $Y(g)$ of a (semi)simple Lie algebra $g$ over $k$ (will also discuss generalization to affine Kac-Moody)

$Y(g)$ - Hopf algebra deformation of

- $U(g[t])$ - non-commutative
- $O(g[[t^{-1}]])$ - co-commutative

both non-comm. and non-co-comm.

Shows up in:
- Classical RT (Branching rules for GL(n) and other classical groups, Gelfand-Tsetlin theory)
- Symplectic/algebraic geometry (quantization of symplectic structures, affine Grassmannian, GRT + Quiver varieties)
- Math. Phys. (Integrable systems, e.g. Toda, Heisenberg)
- Categorification $\Rightarrow$ Algebraic Combinatorics

Plan: 1. Start with motivating examples from classical RT + Gelfand-Tsetlin theory
2. Structure theory of Yangians
3. Representations of Yangians
4. Applications to integrable systems, geometry, and combinatorics

2. Generalities about representations of $GL(n)$
Semisimple category; simples $L(\lambda)$, $\lambda$ highest wt.

Characterization of $L(\lambda)$: $\mathbb{C}^n = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-

\mathfrak{h}_+ = \langle 0, 0, \ldots, 1 \rangle = \text{span} \{ E_{ij} \mid i < j \}
\mathfrak{h}_- = \langle 0, 0, \ldots, -1 \rangle = \text{span} \{ E_{ij} \mid i > j \}
\lambda = (\lambda_1, \ldots, \lambda_N): \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N

L(\lambda) \text{ is generated by } \{ E_i \}_{i \in \mathbb{Z}}: \mathfrak{n}_+ \mathfrak{h}_- \mathfrak{n}_- = 0, E_i \mathfrak{h}_- = \lambda_i \mathfrak{h}_-

Examples:
1) $L(0) = \mathbb{C}$ - trivial
2) tauntological: $\mathbb{C}^N = L(0), \omega_1 = (1, 0, \ldots, 0)$
3) $L^\wedge \mathbb{C}^N = \text{L}(\omega_1), \omega_1 = (1, 1, 0, \ldots, 0)$ - $k$-th fundamental wt $k$
4) $L^\wedge \mathbb{C}^N = \text{L}(\omega_1)$, $S = (1, 1, 1, 1) = \omega_N$

2. $\mathfrak{h}_+$ acts on $\text{det}^g$ trivially on $\text{SL}_N$
$L(\delta) \otimes L(\lambda) = L(\lambda + \delta)$
$L(\delta) \otimes L(\lambda) = L(\lambda - \delta)$

So $\text{SL}_N$ rep. $L(\lambda)$ depends on $\lambda$ up to $\mathbb{Z}$.

3) Schur-Weyl duality + consequences

Idea: $\mathbb{C}^N = L(\omega_1)$ is a faithful $GL_N$ rep.
so if generates $\text{Rep} GL_N$ as a $\otimes$ category
i.e. any $L(\lambda)$ arises as a summand
in some $(\mathbb{C}^N)^m \otimes (\mathbb{C}^N)^m$

We'll show that for $\lambda_i \geq 0 \forall i$, $L(\lambda)$ arises in $(\mathbb{C}^N)^m$, $m = \lambda_1 + \ldots + \lambda_N$, i.e. $\lambda \vdash m$

Consider $M = (\mathbb{C}^N)^m$ $GL_N$ x $S_m$.

($S_m$ permutes the factors)
Let \( H \) be the image of \( U(N) \) in \( \text{End}_A M \) (= image of \( CGT_N \)).

Let \( B \) be the image of \( C_{S_m} \) in \( \text{End}_A M \).

**Thm:** 1) \( \text{End}_A M = B \); 2) \( \text{End}_B M = A \); 3) as a \( G_N \times S_m \)-module, \( M = \bigoplus L(\lambda) \otimes S(\lambda) \)
where \( S(\lambda) \) — Specht module corresponding to \( \lambda \).

**Pf:** Note that (1) \( \Rightarrow \) (2) by the double commutant theorem, since \( B \) is semisimple. Will prove 2):

Have: \( \text{End}_B M = (\text{End}_A M)^{S_m} = \bigoplus (\text{End}_A C^N)^{S_m} = \bigoplus S^m \text{End}_A C^N \)

with symmetric power.

Need to show \( S^m \text{End}_A C^N = \text{span } \{ g \circ \text{mult } | g \in G_N \} \)
But this follows from the general

**Lemma (ex. in linear algebra):**

\( V \) — f.d. vector space / \( C \)

\( U \) — open subset. Then \( S^m V = \text{span } \{ u \circ \text{mult } | u \in U \} \)

3) to show \( M = \bigoplus L(\lambda) \otimes S(\lambda) \), need a characterization of \( S(\lambda) \):

\( \lambda = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow H \in \text{Sym}_3 \) — row subgp (preserves rows)

Any table \( H \in \text{Sym}_n \) — column subgp

Then \( S(\lambda) \) is the only irrep. of \( S_n \)
s.t. \( \exists \upsilon \in S(n) \) s.t. \( \forall h \in H \) \( h \upsilon = \text{sgn}(h) \upsilon \) \( (*) \)

\[
\text{Sym}_H \upsilon := \frac{1}{|H|} \sum_{h \in H} h \upsilon = 0,
\]

b) the following characterization of \( L(\lambda) \):

\( L(\lambda) = \text{highest component in } \bigotimes (N^k C^N)^{\otimes n_k} \)

Where \( \lambda = k \sum h_k \omega_k \), i.e., generated by

\[
\lambda = \bigoplus_{\omega_k} \bigotimes_{n_k}
\]

Using this, have a \( L(\lambda) C \bigotimes (C^N)^{\otimes n} \) generated by a vector \( \psi_\lambda \), skew-symmetric w.r.t. \( H \), and easy check \( \text{Sym}_H \psi_\lambda \neq 0 \) (highest tensor invariant survives) \( \Rightarrow \)

\( \Rightarrow \) the multiplicity space of \( L(\lambda) \) in \( \mathcal{M} \)

contains \( \upsilon \) as in \( (*) \)

Reformulations of SW duality:

Thm (Fundamental theorem of Invariant Theory):

\[
\text{Hom}_{GL_N}( \bigotimes (C^N)^{\otimes p}, C ) = \bigoplus_{p \neq q} \bigoplus \bigoplus \bigoplus
\]

\[
\begin{cases}
0 & \text{if } p \neq q \\
\text{span} \{ \phi_\lambda \sigma \in S_m \} & \text{if } p = q = m
\end{cases}
\]

Thm: \( C[\text{Mat}_N]^{\otimes k} \otimes_{GL_N} \) (polynomial invariants of \( k \)-tuple of \( N \times N \)-matrices under simultaneous conjugation) is generated by
\[(M_1, \ldots, M_k) \rightarrow \text{Tr} \ M_i \cdot M_i^p \quad \text{for} \quad p \in \mathbb{Z}_{\geq 0}, \quad 1 \leq i, \ldots, i \leq k.\]