Multiple Choice Questions

For problems 1-7, no justification is required.

(1) The derivative of function $f(x)$ can be expressed as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$

(a) True  
(b) False

\[ \text{If } \lim_{h \to 0} g(h) \text{ exists, then it is equal to } \lim_{h \to 0} g(-h), \text{ because } h \text{ and } -h \text{ both approach 0} \]

(2) The function $f(x)$ has jump discontinuity at $x = 0$.

$$f(x) = \begin{cases} 
-x + 1 & x > 0 \\
2 & x = 0 \\
1 + 2x & x < 0 
\end{cases}$$

(a) True  
(b) False

\[ \text{Removable, as } \lim_{x \to \infty} f(x) = 1 \neq f(c) \]

(3) What is the average rate of change of $f(x) = \ln x^2$ between $x = 1$ and $x = 2$?

(a) $\ln 4$  
(b) 0  
(c) $\ln 2$  
(d) 1  
(e) 2

\[ \frac{\ln(2^2) - \ln(1^2)}{2-1} = \frac{\ln(4) - 0}{1} \]

(4) If a function is continuous at $x = a$, then it is also differentiable at $x = a$.

(a) True  
(b) False

\[ \text{Counter-example is } f(x) = |x| \text{ at } a = 0 \]
(5) Consider the two graphs of functions

(a) Is \( f(x) \) differentiable at \( x = -2 \)? Why?
(b) Find \( (g \circ f)'(0) \)
(c) Find \( (f \circ g)'(1) \).
(d) Find \( (f \cdot g)'(1) \).

\[ (g \circ f)'(0) = g'(f(0)) \cdot f'(0) \]

No, because the slope is negative at -2 and positive at -2.

\[ = -3 \cdot \text{undefined} \]

Let \( g'(x) = -3 \).

\[ (f \circ g)'(1) = f'(g(1)) \cdot g'(1) \]

\[ = f'(-3) \cdot -3 \]

\[ = -\frac{5}{3} \cdot -3 = 5 \]

\[ (f \cdot g)'(1) = f'(1)g(1) + g'(1)f(1) \] (product rule)

\[ = -1 \cdot -3 + -3 \cdot 1 \]

\[ = 0 \]
Use the following figure to answer questions (6) and (7):

(6) Match the following expressions to their limits by drawing a line. Note, more than one expression can have the same limit.

\[ \lim_{x \to -8} f(x) = -6 \]
\[ \lim_{x \to 2^-} f(x) = -3 \]
\[ \lim_{x \to 6^-} f(x) = 0 \]
\[ \lim_{x \to 10^+} f(x) = 2 \]
\[ \lim_{x \to -4} f(x) \text{ No Limit} \]

(7) Match the following points to their type of discontinuity by drawing a line.

\[ x = -8 \text { Infinite Discontinuity} \]
\[ x = -2 \text { Removable Discontinuity} \]
\[ x = 6 \text { Continuous} \]
\[ x = 10 \text { Jump Discontinuity} \]
\[ x = -4 \]
Long Answer Questions

For problems, please show all your work to receive full credit.

(8) (a) Let $\mathcal{M} = \{\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \ldots\}$

Assuming $a_n$ continues in the same pattern, which of the following is a general form for $a_n$?

(i) $\frac{1}{n}$

(ii) $\frac{n}{2n - 1}$

(iii) $\frac{n + 1}{n + 3}$

(iv) $\frac{n}{n + 2}$

(b) What does $\{a_n\}$ converge to?

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{n + 2} = \lim_{n \to \infty} \frac{n + 2 - 2}{n + 2} = \lim_{n \to \infty} \frac{1}{1 + \frac{2}{n}} = \frac{1}{1 + 0} = 1$$

Limited quotient and sum rules

(c) Let $\epsilon = \frac{1}{4}$. Find an $N$ that satisfies the definition of convergence for this $\epsilon$. Justify.

For all values of $n \geq 7$,

$$\left| \frac{n}{n+2} - 1 \right| = 1 - \frac{n}{n+2} = \frac{n+2-n}{n+2} = \frac{2}{n+2} \leq \frac{2}{9} < \frac{1}{4}$$

So $N = 7$.

I found $7$ by looking for the smallest value $n$ such that $\frac{2}{n+2} < \frac{1}{4}$. Since $\frac{2}{6+2} = \frac{1}{4}$, it is the smallest value of $N$. Any integer greater than $7$ would also work.
(9) Prove that there is at least a number $x$ between 0 and $\pi$ such that $\tan(x) = x + \frac{1}{3}$.

$$\tan(x) = x + \frac{1}{3} \quad \iff \quad 0 = \tan(x) - x - \frac{1}{3}.$$ 

Let $f(x) = \tan(x) - x - \frac{1}{3}$. This function is continuous over the interval $(0, \frac{\pi}{3})$ (actually, it is continuous over a bigger interval, but that is not needed).

Also, $f(0) = -\frac{1}{3}$ and $f\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{\pi}{3} - 1 > \frac{3}{3} - \frac{1}{2} > 0$.

Since $f(0) < 0$ and $f\left(\frac{\pi}{3}\right) > 0$, we know by the intermediate value theorem that there exists a value $x$ between 0 and $\frac{\pi}{3}$ such that $f(x) = 0$.

For that number $x$, $\tan(x) = x + \frac{1}{3}$.
(10) (a) Find the value of $k$ for which $f(x)$ is continuous.

$$f(x) = \begin{cases} x^2 & x < k \\ 2x - 1 & k \leq x \end{cases}$$

We want $k$ such that $k^2 = 2k - 1$. Hence, these are the solutions to $k^2 - 2k + 1 = 0$, so $k=1$ is the only solution.

*This is for whenever $\lim_{x \to k^-} f(x) = f(k)$. The right-limit is always equal to the value since $2x-1$ is a continuous function.*

(b) Is $f(x)$ differentiable at $x = k$? Why or why not?

It is!

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1^-} x + 1 = 2.$$  

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{2x-1}{x-1} = \lim_{x \to 1^+} \frac{2(x-1)}{x-1} = \lim_{x \to 1^+} \frac{2(x-1)}{x-1} = 2.$$  

Since the limits on each side agree, $\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$ is defined, and that is the derivative.
(11) A student forgot to justify the computation of the following limit

\[
\lim_{x \to 3} \frac{\sin(x - 3 + \frac{\pi}{3})}{\cos(-\pi x)}.
\]

Below is the student’s computation. Explain what theorem the student used at each step, and explain if the student respected the conditions for using these theorems.

\[
\begin{align*}
\lim_{x \to 3} \frac{\sin(x - 3 + \frac{\pi}{3})}{\cos(-\pi x)} &= \frac{\lim_{x \to 3} \sin(x - 3 + \frac{\pi}{3})}{\lim_{x \to 3} \cos(-\pi x)} \\
&= \frac{\sin(\lim_{x \to 3} x - 3 + \frac{\pi}{3})}{\cos(\lim_{x \to 3} -\pi x)} \\
&= \frac{\sin(-1)}{\cos(-\pi)} \\
&= -1
\end{align*}
\]

(1) **Quotient rule:** may be used if both limits (in the numerator and denominator) are defined and if the one in the denominator is not 0. (Here, this is \(\lim_{x \to 3} \cos(-\pi x) = -1\).)

(2) The functions \(\sin(x)\) and \(\cos(x)\) are continuous; hence \(\lim_{x \to a} (\sin(x)) = \sin(\lim_{x \to a} x)\) and \(\lim_{x \to a} (\cos(x)) = \cos(\lim_{x \to a} x)\).