Recall: • <u>bilinear form</u> $b: V\times V \to k \longleftrightarrow \varphi_b \in \mathrm{Hom}(V,V^*)$

$\varphi_b(v) = b(v,\cdot) = \left(\begin{smallmatrix} V\to k \\ w \mapsto b(v,w)\end{smallmatrix}\right)$

This gives an isom. $B(V) \simeq \mathrm{Hom}(V,V^*)$.

• b is <u>nondegenerate</u> if φ_b is an isom.

• in a basis $(e_1,\dots,e_n)$, represent b by a matrix A with entries $a_{ij} = b(e_i,e_j)$

$$b(\Sigma x_i e_i, \Sigma y_j e_j) = \Sigma\, a_{ij} x_i y_j \quad = X^T A Y.$$

b is symmetric iff A is symmetric $(a_{ij} = a_{ji})$
nondegenerate iff A is invertible

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* **Def**: If $S\subset V$ is a subspace of a vector space equipped with a bilinear form $b: V\times V\to k$, we define its <u>orthogonal complement</u>
$S^\perp = \{v\in V \,/\, b(v,w) = 0 \ \ \forall w\in S\}$. This is a vector subspace.

* corrected from an earlier* version where v, w were swapped, for consistency with the Lemma below.

⚠ This is most intuitive if b is symmetric or skew. Otherwise we have to worry about "left-orthogonal" vs. "right-orthogonal" to S.

* **Lemma**: If b is nondegenerate then $\dim S^\perp = \dim V - \dim S$ (else $\geq$)

Proof: $S^\perp = \mathrm{Ker}\left(\begin{smallmatrix} V \to S^* \\ v \mapsto \varphi_b(v)|_S\end{smallmatrix}\right)$ composition of $\varphi_b: V\to V^*$ and restriction $r: \begin{smallmatrix} V^* \twoheadrightarrow S^* \\ \ell \mapsto \ell|_S\end{smallmatrix}$

By dim-formula, $\dim S^\perp = \dim V - \mathrm{rank}(r\circ\varphi_b)$. If b is nondegenerate then φ_b isomorphism and r surjective $\Rightarrow \mathrm{rank}(r\circ\varphi_b) = \dim S^* = \dim S$; in general $\leq$ □

Ex: • $V = \mathbb{R}^n$ with the standard dot product $b(v,w) = \sum_{i=1}^n v_i w_i$: then
$V = S\oplus S^\perp$ the "usual" orthogonal complement
because: $S\cap S^\perp = \{0\}$ (see below) and $\dim S + \dim S^\perp = \dim V$.

• but for $b: k^2\times k^2 \to k$
$$b((x_1,x_2),(y_1,y_2)) = x_1 y_2 - x_2 y_1$$ (skew-symmetric, nondegenerate)
$S\subset k^2$ 1-dim.! subspace spanned by any nonzero vector $v \Rightarrow \underline{S^\perp = S!!}$
(because $b(v,w) = 0 \Leftrightarrow w\in k\cdot v = S$).

Inner product spaces:

Def^n: An <u>inner product space</u> is a vector space V over $\mathbb{R}$ together with a <u>symmetric</u> <u>definite positive</u> bilinear form $\langle\cdot,\cdot\rangle: V\times V\to\mathbb{R}$

Symmetric: $\langle u,v\rangle = \langle v,u\rangle$ **Def. positive**: $\langle u,u\rangle \geq 0\ \forall u\in V$, and $\langle u,u\rangle = 0$ iff $u=0$.

This definition only makes sense over an <u>ordered</u> field so "$\langle u,u\rangle \geq 0$" makes sense. In practice this means $\mathbb{R}$. We can't do this over $\mathbb{C}$. (we'll see a workaround: Hermitian forms)

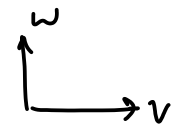
- Let $\varphi : V \to V^*$, $v \mapsto \langle v, \cdot \rangle$ be the linear map corresponding to $\langle \cdot,\cdot \rangle$.

  $\langle \cdot,\cdot\rangle$ definite positive $\Rightarrow$ φ is injective (since $\forall v \neq 0$, $\varphi(v) \neq 0$! $\varphi(v)(v) > 0$).
  $\Rightarrow$ (assuming $\dim V < \infty$) φ is an <u>iso.</u> $V \xrightarrow{\sim} V^*$, ie. $\langle \cdot,\cdot\rangle$ is <u>nondegenerate</u>. (The converse is false: $\langle \cdot,\cdot\rangle$ nondegenerate $\not\Rightarrow$ positive).

<u>Prop:</u> V finite-dim inner product space, $S \subset V$ subspace $\Rightarrow V = S \oplus S^\perp$.

<u>Pf:</u>
- we've seen: $\langle \cdot,\cdot\rangle$ is nondegenerate so $\dim S^\perp = \dim V - \dim S$.
- since $\langle \cdot,\cdot\rangle$ is positive definite, $v \in S \cap S^\perp \Rightarrow \langle v,v\rangle = 0 \Rightarrow v = 0$.
  So $S \cap S^\perp = \{0\}$. Since dimensions add up to $\dim V$, this implies $S \oplus S^\perp = V$. $\square$

<u>Def:</u>
- The <u>norm</u> of a vector is $\|v\| = \sqrt{\langle v,v\rangle}$.
- $v, w \in V$ are <u>orthogonal</u> if $\langle v,w\rangle = 0$.



<u>Observe:</u> $\|v-w\|^2 = \langle v-w, v-w\rangle = \|v\|^2 + \|w\|^2 - 2\langle v,w\rangle$.
$\to$ if v and w are orthogonal then $\|v-w\|^2 = \|v\|^2 + \|w\|^2$ Pythagorean thm
$\to$ in general, by analogy with law of triangles, we define the <u>angle</u> b/w 2 vectors
$$\measuredangle(v,w) = \cos^{-1}\left(\frac{\langle v,w\rangle}{\|v\|\,\|w\|}\right).$$ This only makes sense if $|\langle v,w\rangle| \leq \|v\|\,\|w\|$?

<u>Theorem</u> (<u>Cauchy-Schwarz inequality</u>) $\forall u, v \in V$, $|\langle u,v\rangle| \leq \|u\|\,\|v\|$.

<u>Pf:</u> The inequality is unaffected by scaling so we can assume $\|u\| = 1$.
Decompose v along $V = S \oplus S^\perp$ where $S = \mathrm{span}(u) \subset V$. Explicitly,
$v = v_1 + v_2$, $v_1 = \langle v,u\rangle u \in \mathrm{span}(u)$, $v_2 = v - \langle v,u\rangle u$ orthogonal to u.

Then $v_1 \perp v_2$ so $\|v\|^2 = \|v_1\|^2 + \|v_2\|^2 \geq \|v_1\|^2 = \langle v,u\rangle^2$.
This is the desired inequality for $\|u\| = 1$. $\square$

<u>Def:</u> V finite dim! $/\mathbb{R}$ with inner product $\langle \cdot,\cdot\rangle$. A basis $v_1 \dots v_n$ of V is said to be <u>orthonormal</u> if
$$\langle v_i, v_j\rangle = \begin{cases} 1 & i = j \quad (\text{ie. } \|v_i\| = 1) \\ 0 & i \neq j \quad (\text{ie. } v_i \perp v_j) \end{cases}$$

In such a basis, $(V, \langle \cdot,\cdot\rangle) \cong (\mathbb{R}^n$ with standard dot product$)$.

Thm: Every finite-dimensional inner product space (/$\mathbb{R}$) has an orthonormal basis.

<u>Proof 1</u>: By induction on $\dim(V)$: choose $v \neq 0 \in V$, let $v_1 = \frac{v}{\|v\|}$ so $\|v_1\| = 1$.
Then let $S = \operatorname{span}(v_1)$, $V = S \oplus S^\perp$.
Let $v_2, \dots, v_n$ be an orthonormal basis for $S^\perp$ (the restriction of $\langle \cdot, \cdot \rangle$ to $S^\perp$ is an inner product!)
Then $v_1 \dots v_n$ is an orthonormal basis for V (check!). $\square$

<u>Proof 2</u>: start with any basis $w_1 \dots w_n$ of V and use the <u>Gram-Schmidt process</u>.
First set $v_1 = \frac{w_1}{\|w_1\|}$. Then take $w_2 - \langle w_2, v_1 \rangle v_1$ which is $\perp v_1$
(and nonzero by independence of w_i), set $v_2 = \dfrac{w_2 - \langle w_2, v_1 \rangle v_1}{\| w_2 - \langle w_2, v_1 \rangle v_1 \|}$

and so on, set $v_j = \dfrac{w_j - \sum_{i=1}^{j-1} \langle w_j, v_i \rangle v_i}{\left\| w_j - \sum_{i=1}^{j-1} \langle w_j, v_i \rangle v_i \right\|}$. Then $(v_1, \dots, v_n)$ is an orthonormal basis $\square$

So: every finite dim! inner product space /$\mathbb{R}$ is isomorphic (as an inner product space, not just as a vector space) to standard $\mathbb{R}^n$, $n = \dim V$.

<u>Operators on inner product spaces</u>: Let $(V, \langle \cdot, \cdot \rangle)$ inner product space. There are two special classes of linear operators on V of interest to us.

<u>Def</u>: Say $T: V \to V$ is an <u>orthogonal operator</u> if it respects the inner product, ie. $\langle Tu, Tv \rangle = \langle u, v \rangle \ \forall u, v \in V$.

(In other terms, T "preserves lengths and angles").

<u>Remarks</u>: 1) orthogonal operators map orthonormal bases to orthonormal bases!

$$\langle Te_i, Te_j \rangle = \langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

in particular, orthogonal operators are always invertible!

2) If T is orthogonal then T^{-1} is orthogonal

$$\left(\langle T^{-1}u, T^{-1}v \rangle \underset{\uparrow T \text{ orthogonal}}{=} \langle T(T^{-1}u), T(T^{-1}v) \rangle = \langle u, v \rangle \ \forall u, v \right)$$

If T_1, T_2 are orthogonal then so is $T_1 T_2$ (check!)

<u>Hence</u>: orthogonal operators form a <u>subgroup of $\mathrm{Aut}(V)$</u>.

3) If M is the matrix representing T in an orthonormal basis, then $M^T M = I$.

Indeed: entries of $M^T M$ = dot products of columns of M!

$$(M^TM)_{ij} = \sum_k M^T_{ik} M_{kj} = \sum_k M_{ki} M_{kj} = \langle M(e_i), M(e_j)\rangle = \langle e_i, e_j\rangle.$$

<u>Def</u>: Let $T: V \to V$ linear operator on an inner product space $(V, \langle \cdot,\cdot\rangle)$. There exists a unique linear operator $T^*: V \to V$, called the <u>adjoint</u> of φ, such that $\langle v, T(w)\rangle = \langle T^*(v), w\rangle$ $\forall v, w \in V$.

<u>Indeed</u>: given $v \in V$, the linear functional
$$\begin{aligned} V &\longrightarrow \mathbb{R} \\ w &\longmapsto \langle v, T(w)\rangle \end{aligned}$$
is, using nondegeneracy of $\langle \cdot,\cdot\rangle$, given by the inner product of w with some element of V, which we call $T^*(v)$; then check this has linear dependence on v.

<u>Equivalently</u>: $\langle \cdot,\cdot\rangle$ defines an isom. $\varphi: V \xrightarrow{\sim} V^*$. Then T^* is the composition of

$$\begin{array}{ccccccc} V & \xrightarrow{\varphi} & V^* & \xrightarrow{T^t} & V^* & \xrightarrow{\varphi^{-1}} & V \\ v & \longmapsto & \langle v, \cdot\rangle & \longmapsto & \langle v, T(\cdot)\rangle = \langle T^*(v), \cdot\rangle & \longmapsto & T^*(v). \end{array}$$

<u>Def</u>: $T: V \to V$ is <u>self-adjoint</u> if $T^* = T$. (ie. $\langle v, Tw\rangle = \langle Tv, w\rangle$ $\forall v, w$).

* In an orthonormal basis $(e_1, \dots, e_n)$ of V, $\langle v, w\rangle = v^t w$, so

transpose gives a row vector; column vector

if matrix of T is M, T^* is N,

$$\left.\begin{aligned} \langle v, T(w)\rangle &= v^t M w \\ \langle T^*(v), w\rangle &= (Nv)^t w = v^t N^t w \end{aligned}\right\} \Rightarrow \text{comparing: } N^t = M, \text{ so } N = M^t.$$

<u>Hence</u>: $M(T^*) = M(T)^t$ in orthonormal basis; T is self-adjoint $\Leftrightarrow$ $M(T)$ <u>symmetric</u>

Note that self-adjoint operators ($\sim$ symmetric matrices) need not be invertible. For example 0 is a self-adjoint operator...

<u>Prop</u>: if T is self-adjoint and $S \subset V$ is an invariant subspace ($T(S) \subset S$) then $S^\perp$ is also an invariant subspace ($T(S^\perp) \subset S^\perp$)

<u>Pf</u>: Let $v \in S^\perp$, then $\forall w \in S$, $T(w) \in S$, so $\langle Tv, w\rangle \underset{(T^* = T)}{=} \langle v, Tw\rangle \underset{(v \in S^\perp, Tw \in S)}{=} 0$.
Since $\langle Tv, w\rangle = 0$ $\forall w \in S$, we get: $Tv \in S^\perp$. $\square$

<u>Theorem</u> (The <u>spectral theorem</u> for real self adjoint operators)

If $T: V \to V$ is self adjoint then T is diagonalizable, with real eigenvalues. Even more, T can be diagonalized in an orthonormal basis of $(V, \langle \cdot,\cdot\rangle)$!