Last time we saw how rep's of S_3 can be decomposed into irreducibles efficiently by looking at eigenspaces of the transformations by which certain elements of S_3 act.

<u>Recall:</u> the irred. representations of S_3 are
- trivial rep. $U=\mathbb{C}$, σ acts by 1
- alternating $U'=\mathbb{C}$ $\quad (-1)^\sigma$
- standard $V=\{z_1+z_2+z_3=0\}\subset\mathbb{C}^3$, σ permutes coords

and in terms of action of $\tau=$ 3-cycle, $\sigma=$ transposition
$$\begin{cases} U: & \tau=id \quad \sigma=id \\ U' & \tau=id \quad \sigma=-id \\ V & \tau\sim\begin{pmatrix}\lambda&0\\0&\lambda^2\end{pmatrix},\ \sigma\sim\begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \lambda=e^{2\pi i/3}\end{cases}$$

$\Rightarrow$ given any rep^n W of S_3, $W\simeq U^{\oplus a}\oplus U'^{\oplus b}\oplus V^{\oplus c}$,
the $+1$-eigenspace of $\tau: W\to W$ has dim. $a+b$, λ/λ^2-eigenspaces dim. c
the $+1$-eigenspace of σ has dim. $a+c$, -1-eigenspace dim. $b+c$.

<u>Example:</u> consider V the standard rep. of S_3, and $V^{\otimes 2}=V\otimes V$ also a rep^n (recall: $g(v\otimes w)=gv\otimes gw$). How does $V^{\otimes 2}$ decompose into irreducibles?

Start with a basis e_1, e_2 of V with $\tau e_1=\lambda e_1$, $\tau e_2=\lambda^2 e_2$ where $\lambda=e^{2\pi i/3}$
$$\sigma e_1=e_2,\quad \sigma e_2=e_1.$$

Then $V\otimes V$ has a basis $e_1\otimes e_1,\ e_1\otimes e_2,\ e_2\otimes e_1,\ e_2\otimes e_2$.
These are eigenvectors of τ, with eigenvalues $\lambda^2,\ 1,\ 1,\ \lambda$.

Moreover, on the 1-eigenspace $\mathrm{span}(e_1\otimes e_2, e_2\otimes e_1)$, σ swaps these two, so
$e_1\otimes e_2\pm e_2\otimes e_1$ is an eigenvector of σ with eigenvalue ± 1.

Hence $V\otimes V\simeq U\oplus U'\oplus V$.

Similarly $\mathrm{Sym}^2 V$: basis $e_1^2,\ e_1e_2,\ e_2^2$ $\rightsquigarrow \mathrm{Sym}^2(V)\simeq U\oplus V$.
τ acts by $\lambda^2,\ 1,\ \lambda$

(whereas $\Lambda^2 V\simeq U'$, perhaps unsurprisingly considering det. vs sign).

This generalizes to more complicated groups - we'll see that eigenvalues go a long way towards classifying representations - but we need some way of organizing the information.

<u>Digression</u>: <u>Symmetric polynomials</u>: (this is all <u>motivation</u> for the study of characters).

- <u>Observe</u>: an efficient way to store information about n (complex) numbers, unordered and possibly with repetitions, is to specify the coefficients of the polynomial of which they are the roots, ie. $\prod_{i=1}^n (x-\lambda_i)$. These coefficients are <u>symmetric polynomials</u> in $\lambda_1\dots\lambda_n$

- S_n acts on the space of polynomials $\mathbb{C}[z_1,\dots,z_n]$ by permuting the variables.

Def: A <u>symmetric polynomial</u> is $f \in \mathbb{C}[z_1,\dots,z_n]$ that is a fixed point of the S_n-action, $\sigma(f) = f \quad \forall \sigma \in S_n$.

(<u>Rmk</u>: equality of polynomials means, as usual, equality of coefficients, which over a finite field is a stronger condition than having equality as functions on k^n. Of course over $\mathbb{C}$ no difference.)

Def: The <u>elementary symmetric polynomials</u>: $\sigma_1(z_1,\dots,z_n) = \sum_{i=1}^n z_i$,

$$\sigma_2(z_1,\dots,z_n) = \sum_{1\le i<j\le n} z_i z_j, \quad \dots, \quad \sigma_k = \sum_{1\le i_1<\dots<i_k\le n} z_{i_1}\cdots z_{i_k}, \quad \dots \quad \sigma_n = \prod_{i=1} z_i.$$

<u>Check</u>: the coefficient of x^{n-k} in $\prod_{i=1}^n (x - z_i)$ is, up to sign $(-1)^k$, $\sigma_k(z_1,\dots,z_n)$.

<u>Hence</u>: the fundamental theorem of algebra gives a bijection

$$\{\text{unordered } n\text{-tuples of complex numbers, repetitions allowed}\} \overset{\sim}{\longleftrightarrow} \underset{\text{ordered tuples}}{\mathbb{C}^n}$$

$$[z_1,\dots,z_n] \longmapsto (\sigma_1(z_i),\dots,\sigma_n(z_i))$$

$$[\text{the roots of } x^n - \sigma_1 x^{n-1} + \dots + (-1)^n \sigma_n] \longleftarrow (\sigma_1,\dots,\sigma_n)$$

In other terms: $[z_1,\dots,z_n] \longleftrightarrow$ coefficients of the polynomial $\prod(x-z_i)$.

<u>Theorem</u>: The subring of symmetric polynomials in $\mathbb{C}[z_1\dots z_n]$, ie. $\mathbb{C}[z_1\dots z_n]^{S_n}$, is isomorphic to the polynomial algebra in n variables $\mathbb{C}[\sigma_1,\dots,\sigma_n]$.
Ie. every symmetric polynomial is uniquely a polynomial expression in the elementary symmetric polynomials.

* We won't prove this, but to see why this works, look at the case $n=2$.
The vector space of symmetric polynomials has basis

$$1 = 1$$
$$z_1 + z_2 = \sigma_1$$
$$z_1^2 + z_2^2 = (z_1+z_2)^2 - 2z_1z_2 = \sigma_1^2 - 2\sigma_2$$
$$z_1 z_2 = \sigma_2$$

$$z_1^3 + z_2^3 = \sigma_1^3 - 3z_1^2z_2 - 3z_1z_2^2 = \sigma_1^3 - 3\sigma_1\sigma_2$$
$$z_1^2 z_2 + z_1 z_2^2 = \sigma_1\sigma_2$$
$$\dots\dots$$

Observe: any symmetric polynomial in 2 variables can be written as

$$\begin{aligned} p(z_1,z_2) &= \sum a_k (z_1^k + z_2^k) + z_1 z_2\, q(z_1,z_2) \\ &= \sum a_k (z_1+z_2)^k + z_1 z_2\, q'(z_1,z_2) \\ &= \sum a_k \sigma_1^k + \sigma_2 . q' \end{aligned}$$
& work by induction on degree.

Rmk: the theorem can be understood in terms of rep theory of S_n!
Namely, the space of homogeneous deg. 1 polynomials is $W_1 = \mathrm{span}(z_1, \dots, z_n) \simeq \mathbb{C}^n$ on which S_n acts by permutation rep. $\cong V \oplus U$ (standard $\oplus$ trivial) and the invariant part is $W_1^{S_n} \simeq U$ trivial summand. Now, homogeneous deg. d polynomials are $W_d = \mathrm{Sym}^d(W_1)$, and the invariant part $W_d^{S_n}$ = trivial summands in the decomp. of W_d into irreducibles! (Unfortunately we haven't studied rep^{ns} of S_n in enough depth to carry through with a proof along these lines).

* Another family of symmetric polynomials are the power sums:

$$\tau_k(z_1, \dots, z_n) = \sum_{i=1}^n z_i^k. \qquad \tau_1 = \sigma_1, \quad \tau_2 = \sigma_1^2 - 2\sigma_2, \ \dots$$

These make sense for all k, but in fact $\tau_1, \dots, \tau_n$ suffice:

Thm': $\mathbb{C}[z_1, \dots, z_n]^{S_n} \cong \mathbb{C}[\tau_1, \dots, \tau_n]$

In particular: specifying an unordered tuple $\{z_1 \dots z_n\}$ is equivalent to specifying $\sum z_i, \sum z_i^2, \dots, \sum z_i^n$.

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* Back to representation theory — why we care about this:
We've seen that, to understand a representation V of G, we should look at the eigenvalues of $g: V \to V$ for each $g \in G$; but this is a lot of information.
We've just said: to specify the eigenvalues λ_i of $g: V \to V$, it is enough to specify the power sums $\sum \lambda_i^k$. But in fact $\sum \lambda_i^k = \mathrm{tr}(g^k)$!
So it's enough to describe just the sum of the eigenvalues $\sum \lambda_i = \mathrm{tr}(g)$ for every $g \in G$ — since G is a group, the trace of g^k is also part of this.

Def: The character χ_V of a representation V is the function $\chi_V : G \to \mathbb{C}$, $\chi_V(g) = \mathrm{tr}(g)$.

Remark: for a 1-dim^l representation of G, ie. a homom. $G \to \mathbb{C}^*$, the character is just the same thing, hence a (multiplicative) homom.
For a higher-dim^l representation, though, $\chi(g_1 g_2) \neq \chi(g_1)\chi(g_2)$.

However, since trace is conjugation invariant, $\mathrm{tr}(ghg^{-1}) = \mathrm{tr}(h)$, so $\chi_V(g)$ only depends on the conjugacy class of g.

Def: A class function $f: G \to \mathbb{C}$ is a function invariant under conjugation, $f(ghg^{-1}) = f(h)$.

<u>Ex:</u> given representations V and W:

- $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$ (eigenvalues of $\left(\begin{array}{c|c}\varphi & 0\\ \hline 0 & \psi\end{array}\right)$ ...)
- $\chi_{V\otimes W}(g) = \chi_V(g)\,\chi_W(g)$ (eigenvalues of $\varphi\otimes\psi : v_i\otimes w_j \mapsto \lambda_i\lambda_j'\, v_i\otimes w_j$)
- $\chi_{V^*}(g) = \overline{\chi_V(g)}$ since g acts by ${}^t(g^{-1})$, and eigenvalues are roots of unity so $\lambda_i^{-1} = \overline{\lambda_i} \Rightarrow \sum \lambda_i^{-1} = \sum \overline{\lambda_i}$
- $\chi_{\Lambda^2 V}(g) = \sum_{i<j} \lambda_i\lambda_j = \frac{1}{2}\left((\sum\lambda_i)^2 - \sum\lambda_i^2\right) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right)$

<u>Ex:</u> If G acts on a finite set S, then there is an associated permutation representation V of dimension $|S|$, with basis $(e_s)_{s\in S}$, G acts by permutation matrices $g\cdot e_s = e_{g\cdot s}$.
Then $\chi_V(g) = tr(g) = \#\{s\in S \,/\, g\cdot s = s\}$, since 1's on diagonal of matrix correspond to fixed points of g, and 0's otherwise.

The <u>character table</u> of a group = list, for each irred. rep^{n} of G, the values of ~~the~~ its character on each conjugacy class of G.

<u>Example:</u> $G = S_3$:

	e	(12)	(123)
U	1	1	1
U'	1	-1	1
V	2	0	-1

→ conjugacy classes

irred. rep^{ns}

$\chi_V(e) = tr(id) = \dim V$.

either from eigenvalues ± 1 for (12), $e^{\pm 2\pi i/3}$ for (123)
or $U\oplus V$ = permutation representation takes values #fixed pts = $(3,1,0)$ then subtract $\chi_U = (1,1,1)$.

Now we have a faster way of decomposing $V\otimes V$ into irreducibles:
$\chi_{V\otimes V}(g) = \chi_V(g)^2$ so $\chi_{V\otimes V}$ takes values $(4,0,1)$
$\chi_U, \chi_{U'}, \chi_V$ are linearly independent, $\chi_{V\otimes V} = \chi_U + \chi_{U'} + \chi_V \Rightarrow V\otimes V \simeq U \oplus U' \oplus V$.
(This is equivalent to counting eigenvalues as we did last time, but somewhat faster!)

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* Now for some magic with characters...

- If V is a representation of G, the invariant part is $V^G = \{v\in V \,/\, gv = v \ \forall g\in G\}$.

<u>Prop:</u> $\varphi = \frac{1}{|G|}\sum_{g\in G} g : V\to V$ is a projection onto $V^G \subset V$: $\begin{cases} Im(\varphi) = V^G \\ \varphi_{|V^G} = id. \end{cases}$

- <u>So:</u> $\dim(V^G) = tr(\varphi) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)$.
- If V, W are rep^{s} of G, $Hom_G(V,W) = Hom(V,W)^G = (V^*\otimes W)^G$, so:

G-action on $Hom(V,W)$ is $g(\varphi) = g\varphi g^{-1}$.

$\dim Hom_G(V,W) = \frac{1}{|G|}\sum_{g\in G} \chi_{V^*\otimes W}(g) = \frac{1}{|G|}\sum_g \overline{\chi_V(g)}\chi_W(g)$ ... more next time.