Math 55a Homework 1

Due Wednesday September 9, 2020.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Extensions will be granted when circumstances genuinely warrant it, but should be requested ahead of the deadline (e-mail Prof. Auroux).
- Questions marked * are harder. It's perfectly ok to have no idea how to get started on those on your own.
- You are encouraged to discuss the homework problems with other students. Start using the `#homework` channel on Slack to ask questions, or `#studygroups` to connect with others and work on a problem together. You should also plan on attending one or more discussion section(s) and office hours in order to ask any questions!
- Handwritten submissions are welcome, but if you have spare time, you could learn how use LaTeX! Sign up for Overleaf at `https://www.overleaf.com/edu/harvard` and see the tutorials at `https://www.overleaf.com/learn`

Material covered: Set theory review (Halmos, chapters 4-10); groups, subgroups, homomorphisms (Artin 2.1-2.6).

0. (a) If you haven't done so already, please post a short self-introduction on the `#general` Slack channel. Consider including your name, where you are from, whether you are on campus, and a hobby or a fun fact about you.

(b) Sometime over the weekend of September 5-7, please complete the week 1 feedback survey (in Canvas). This is important to help us assess how well the course structure, pacing, and our efforts at getting students to know each other are working. (There will be more surveys).

1. Let $f : X \to Y$ be a map of sets. A function $g : Y \to X$ is called a left (resp. right) inverse of f if $g \circ f = \mathrm{id}_X$ (resp. $f \circ g = \mathrm{id}_Y$).

(a) Show that f admits a right inverse if and only if f is surjective (*onto* in Halmos' terminology). Show that, in general, a right inverse is not unique.

(b) Show that f admits a left inverse if and only if f is injective (*one-to-one* in Halmos' terminology). Show that, in general, a left inverse is not unique.

2. Give an explicit bijection between $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$.

3*. Let F denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$, and let $C \subset F$ denote the subset of all continuous functions. Prove that $|\mathbb{R}| = |C| < |F|$. (Hint: use the fact that a continuous function on $\mathbb{R}$ is determined by its values on the rational numbers $\mathbb{Q} \subset \mathbb{R}$.)

4. Let G be a group, and let $x_1, \ldots, x_n \in G$ be any elements. Show that

$$(x_1 x_2 \ldots x_n)^{-1} = x_n^{-1} \cdot x_{n-1}^{-1} \cdots x_1^{-1}.$$

5. Show that a group G cannot be the union of two proper subgroups.

6. Show that any finite group G of even order contains an element $x \in G$ such that $x \neq e$ but $x^2 = e$.

7. Let D_4 be the group of symmetries of a square (including reflections). How many subgroups (including D_4 and $\{e\}$) does D_4 have?

8. Let G be a group, and consider the set map $\phi : G \to G$ sending each element $x \in G$ to its square $\phi(x) = x^2 \in G$. Show that ϕ is a homomorphism if and only if G is abelian.

9. Let G be a group.

(a) Show that the set of automorphisms of G is itself a group (with group law given by composition). This group is denoted $Aut(G)$.

(b) For each element $a \in G$, define a map $c_a : G \to G$ by $c_a(x) = axa^{-1}$. Show that c_a is an automorphism of G.

(c) Show that the map $\phi : G \to Aut(G)$ defined by sending $a \in G$ to $c_a \in Aut(G)$ is a homomorphism.

(d) Give an example of a group G (other than the trivial group $\{e\}$) such that ϕ is an isomorphism.

10*. Let S be a nonempty finite set, equipped with an associative operation $* : S \times S \to S$ such that, for every $x, y \in S$, there exists $z \in S$ such that $x * z = y$, and there exists $z' \in S$ (possibly different from z) such that $z' * x = y$. Show that $(S, *)$ is a group.

11*. (Optional, extra credit) The *free group* on n generators $a_1, \ldots, a_n$, denoted F_n, is the collection of all *reduced words* $a_{i_1}^{n_1} a_{i_2}^{n_2} \ldots a_{i_k}^{n_k}$ of any length $k \geq 0$, where $i_1, \ldots, i_k \in \{1, \ldots, n\}$, $i_j \neq i_{j+1}$, and $n_1, \ldots, n_k$ are non-zero integers, with the law of composition given by juxtaposition and simplification (non-reduced words where $i_j = i_{j+1}$ for some j or some n_j is zero are simplified to reduced ones, by combining repeated terms and eliminating unnecessary ones); the identity is the empty word of length $k = 0$.

(a) Show that there exists an injective homomorphism $F_3 \hookrightarrow F_2$.

(b) Show that for any n there exists an injective homomorphism $F_n \hookrightarrow F_2$.

(c) Show that F_2 is not isomorphic to F_3 – despite the existence of injective homomorphisms $F_2 \hookrightarrow F_3$ (obvious) and $F_3 \hookrightarrow F_2$ (from (a)) (so: group homomorphisms are different from maps of sets, for which the existence of injective maps in both directions implies that of a bijection).

This problem will come back in Math 55b, when we discuss fundamental groups of topological spaces!

12. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?