Today: analogue of inner products for complex vector spaces: Hermitian inner products

As previously noted, a bilinear form on a complex vector space $V \times V \to \mathbb{C}$ can't be definite positive, since $b(iv, iv) = -b(v,v)$. Solution: abandon $\mathbb{C}$-linearity in one of the two variables, and only require "conjugate linear"

Def: A Hermitian form on a complex vector space V is $H: V \times V \to \mathbb{C}$ st.
H is sesquilinear:
- $H(u+v, w) = H(u,w) + H(v,w)$, $H(u, v+w) = H(u,v) + H(u,w)$.
- $H(u, \lambda v) = \lambda H(u,v)$, however $H(\lambda u, v) = \bar{\lambda} H(u,v)$
  ↳ conjugate: $\overline{a+ib} = a - ib$.

$+ H$ conjugate-symmetric: $H(u,v) = \overline{H(v,u)}$.

Conjugate symmetry $\Rightarrow H(u,u) \in \mathbb{R}$.

Def: A Hermitian inner product is a positive definite (conjugate-symmetric) Hermitian form.
↳ ie. $H(u,u) \geq 0 \ \forall u$, $H(u,u) = 0 \Leftrightarrow u = 0$.

Rmk: • $\varphi_H : \begin{array}{l} V \to V^* \\ u \mapsto H(u, \cdot) \end{array}$ is now a complex antilinear map $V \to V^*$! $(\varphi(\lambda u) = \bar{\lambda} \varphi(u))$.

Still, various things carry over from the real case:

- H positive definite $\Rightarrow$ H nondegenerate (ie. $\operatorname{Ker} \varphi_H = 0$)
- Given a subspace $W \subset V$, its orthogonal $W^\perp = \{v \in V \,/\, H(v,w) = 0 \ \forall w \in W\}$ is also a subspace, $V = W \oplus W^\perp$. ($\mathbb{C}$-antilinearity doesn't affect $W^\perp$ being a $\mathbb{C}$-subspace; positive definite implies $W \cap W^\perp = \{0\}$).
- Def: An orthonormal basis of V with a Hermitian inner product is a basis $\{e_i\}$ such that $H(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else.} \end{cases}$

Thm: V admits an orthonormal basis

Same proof as in real case (by induction on $\dim V$: first pick v_1 with $\|v_1\|^2 = H(v_1, v_1) = 1$, then take an orthonormal basis $v_2 \dots v_n$ of $\operatorname{span}(v_1)^\perp$) (or Gram-Schmidt ...).

Corollary: Every finite dim. Hermitian inner product space is isomorphic to $\mathbb{C}^n$ with the standard Hermitian inner product, $H(z,w) = \sum_j \bar{z}_j w_j$.

In matrix form: $H(z,w) = z^* w$ where $z^* = \bar{z}^T = (\bar{z}_1 \dots \bar{z}_n)$ conjugate transpose.

Not-quite-example (Fourier series) $V = C^\infty(S^1, \mathbb{C})$ infinitely differentiable functions $S^1 \simeq \mathbb{R}/\mathbb{Z} \to \mathbb{C}$

def. $\langle f, g\rangle = \int_{S^1} \bar{f}(t)\, g(t)\, dt$ ($\Leftrightarrow$ 1-periodic functions $\mathbb{R} \to \mathbb{C}$)

then $f_n(t) = e^{2\pi i n t}$ are orthogonal, $\langle f_n, f_m\rangle = \delta_{m,n}$.

$\{f_n\}_{n\in\mathbb{Z}}$ not a basis of V, their span $W \subset V$ = space of trigonometric polynomials.

Can think of Fourier series as orthogonal projection onto W.
(Will make more sense with some analysis ... or even better, Hilbert spaces)

- **Def:** V complex vect. space, H Hermitian inner product, $T: V \to V$
  - the *adjoint* of T is $T^*: V \to V$ st. $H(T^* v, w) = H(v, Tw)$ $\forall v, w$ ($\Leftrightarrow H(Tv, w) = H(v, T^* w)$ $\forall v, w$)
  - T is *self-adjoint* if $T^* = T$, ie. $H(v, Tw) = H(Tv, w)$ $\forall v, w \in V$
  - T is *unitary* if $H(Tv, Tw) = H(v, w)$ $\forall v, w \in V$ ie. $T^* = T^{-1}$.

- Unitary operators form a *subgroup* $U(V, H) \subset \mathrm{Aut}(V)$ $(U(n) \subset GL(n, \mathbb{C}))$
  Note $U(1) \simeq S^1$ (multiplication by any complex number of norm 1).
- Note: in an orthonormal basis, $\mathcal{M}(T^*) = \mathcal{M}(T)^* \left(= \overline{\mathcal{M}(T)}^t\right)$.
  This is because $H(Tv, w) = (Mv)^* w = v^* M^* w = H(v, T^* w)$ ✓.
  So: self-adjoint complex operators are described by *Hermitian matrices*, $a_{ij} = \overline{a_{ji}}$.

The complex spectral theorem:

> V finite-dim! complex vector space, $H: V \times V \to \mathbb{C}$ Hermitian inner product, $T: V \to V$ self-adjoint ($T^* = T$) or unitary ($T^* = T^{-1}$), then there exists an orthonormal basis consisting of eigenvectors of T, ie. T is diagonalizable, with eigenvalues $\in \mathbb{R}$ if self-adjoint, $\in S^1$ (unit circle) if unitary.

Proof. As in the real case, the key observation is: if $S \subset V$ is invariant ($T(S) \subset S$) then so is $S^\perp \subset V$. Indeed: in both cases, if S is invariant for T then it is also invariant for $T^* = T^{\pm 1}$. So, if $v \in S^\perp$ then $\forall w \in S$, $H(Tv, w) = H(v, T^* w) = 0$.
So: start with an eigenvector v_1, $Tv_1 = \lambda_1 v_1$, $\|v_1\| = 1$, then let $S = \mathrm{span}(v_1)$ & consider $T_{|S^\perp}$.

★ Back to (not necess. definite) <u>nondegenerate symmetric bilinear forms:</u>

Suppose V is a finite-dimensional vector space over k and $B: V\times V \to k$ is a nondegenerate symmetric bilinear form. Can we classify such B?

(<u>Rmk:</u> $Q(v) = B(v,v) : V \to k$ is something called a <u>quadratic form</u>
can recover B from Q if $\mathrm{char}(k) \neq 2$: $B(u,v) = \frac{1}{2}\left(Q(u+v) - Q(u) - Q(v)\right)$).

<u>Classification approach:</u>
find some vector v s.t. $B(v,v) \neq 0$, and then look at $\mathrm{span}(v)^\perp$
($\mathrm{span}(v)^\perp = \ker(\varphi_B(v) : V \to k)$, so $V = \mathrm{span}(v) \oplus \mathrm{span}(v)^\perp$ when $B(v,v) \neq 0$).
Then study $B_{|\mathrm{span}(v)^\perp}$ ...

⚠ Hermitian forms are what most "normal" people care about, however.

<u>Prop:</u> Over $\mathbb{C}$, any nondegenerate symmetric bilinear form admits a basis $e_1 \dots e_n$ st. $B(e_i, e_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

<u>Proof:</u>
- since $B(u,v) \neq 0 \Rightarrow$ one of $B(u,u)$, $B(v,v)$, $B(u+v,u+v)$ nonzero,
  B nonzero implies the existence of v st. $B(v,v) \neq 0$.
- let $e_1 = B(v,v)^{-1/2}\, v$. Then consider $\mathrm{span}(e_1)^\perp = W$.
  $\mathrm{span}(e_1) \cap \mathrm{span}(e_1)^\perp = \{0\}$ since $B(e_1, e_1) \neq 0$, and
  $\dim W = \dim \ker B(e_1, \cdot) = \dim V - 1 \Rightarrow V = \mathrm{span}(e_1) \oplus W$.
- The restriction of B to W is nondegenerate because the matrix of B in basis $\{e_1$, some basis of $W\}$ is invertible (rank n) iff $B_{|W}$ invertible (rank $n-1$). $\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & B_{|W} \end{array}\right)$
- Complete the proof by induction on dimension (assuming result holds in dim. $n-1$, take e_1 + basis of W st. $B_{|W}(e_j, e_k) = \delta_{jk}$). □

<u>Prop:</u> Over $\mathbb{R}$, any nondegenerate symmetric bilinear form admits a basis st. $B(e_i, e_j) = \begin{cases} 0 & i \neq j \\ \pm 1 & i = j. \end{cases}$

ie. can assume $B\left(\sum_{i=1}^n x_i e_i, \sum_{i=1}^n y_i e_i\right) = \sum_{i=1}^k x_i y_i - \sum_{i=k+1}^n x_i y_i$. $\quad B = \begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{k} & \\ & \overbrace{-1 \ \ddots \ -1}^{n-k} \end{pmatrix}$

We say B has <u>signature</u> $(k, n-k)$. (Case $(n,0)$ = def. positive).
(k = max dim. of a subspace s.t. $B_{|W}$ definite positive, $n-k$ = —"— def. negative).

<u>Proof</u> same as in complex case, except can't always scale to $B(e_1,e_1) = 1$, instead we can only force $B(e_1,e_1) = \pm 1$.

- Over $\mathbb{Q}$, things get <u>much</u> harder – number theory enters!

<u>Ex:</u> $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad Q(v) = B(v,v) = v_1^2 + v_2^2$

$\nexists\, v = (v_1, v_2) \in \mathbb{Q}^2$ st. $B(v,v) = v_1^2 + v_2^2 = 3$

↳ clearing denominators, get $n_1^2 + n_2^2 = 3m^2$

$n_1, n_2, m \in \mathbb{Z}$ no common factor (esp not all even)

However $n_1^2 + n_2^2 \equiv 0, 1, 2 \bmod 4 \quad 3m^2 \equiv 0, 3 \bmod 4$

$\Rightarrow$ necessarily all are even, contradiction.

whereas $B' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ does have $\exists v$ st. $B'(v,v) = 3 \quad (v = (1,1))$

- What about the <u>skew-symmetric</u> case? (suppose $\mathrm{char}(k) \neq 2$)

We can still find a "<u>standard basis</u>" for V finite dim. vect. space with $B : V \times V \to k$ nondegenerate skew-symmetric bilinear form (aka: symplectic form) but the process is slightly different since $B(v,v) = 0 \ \forall v \in V$.

Instead: pick any nonzero $e_1 \in V$; since B is nondegenerate, $B(e_1, \cdot) : V \to k$ is nonzero $\Rightarrow \exists f_1 \in V$ st. $B(e_1, f_1) \neq 0$, can make it $= 1$ by scaling f_1.

Now we find $\mathrm{span}(e_1, f_1) \cap \mathrm{span}(e_1, f_1)^{\perp} = \{0\}$ (if $v = ae_1 + bf_1$ has $B(v, e_1) = B(v, f_1) = 0 \Rightarrow a = b = 0$)

so $V = \mathrm{span}(e_1, f_1) \oplus \mathrm{span}(e_1, f_1)^{\perp}$,

and study the restriction of B to the latter subspace (induction on dim.).

$\Rightarrow$ <u>Prop:</u> V finite dim! over k, $\mathrm{char}(k) \neq 2$,
B nondegenerate skewsymmetric bilinear form $V \times V \to k$
$\Rightarrow \dim V$ is even, and V has a basis $(e_1, f_1, \dots, e_n, f_n)$ st.
$B(e_i, e_j) = B(f_i, f_j) = 0, \quad B(e_i, f_j) = \delta_{ij} = -B(f_j, e_i)$.

ie. matrix of B is
$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \\ & & \ddots \end{pmatrix}$$

The group of linear transformations preserving B is called the <u>symplectic group</u> $Sp(V, B) \simeq Sp(2n, k)$.

<u>Next time:</u> tensor product & multilinear algebra. This gives us a way to think of bilinear (or <u>multilinear</u>) maps $V_1 \times V_2 \to W$ as linear maps from a new vector space $V_1 \otimes V_2$.