1. Comment about \( S_\lambda(x) = \sum_T x^{\mu(T)} \) for how to prove this is symmetric.

Combinatorial argument: there are Bender-Knuth involutions \( t_i : GT(\lambda) \to GT(\lambda) \) such that \( \mu(t_i T) = s_i \mu(T) \) if \( i \neq \lambda^*_1 \).

**Definition:** \( t_i T = \hat{T} = \sqrt{\lambda^*_i} \sqrt{\lambda_i} \), \( \lambda_i = \left\{ \begin{array}{ll} \lambda^*_i, & k = i \\ \min(\lambda^*_j, \lambda^*_i) + \max(\lambda^*_i, \lambda^*_j) & k \neq i \end{array} \right. \)

(i.e. \( t_i \) reverses the range for \( \lambda_i \) for all \( j = 1, \ldots, i \)).

**Remark:** Can regard \( GT(\lambda) \) as a convex polytope, then \( t_i \) are some PL transformations of \( GT(\lambda) \).

**Proposition:** \( \mu(t_i T) = s_i \mu(T) \). **Proof:** Straightforward check \( \mu_{i+1} \mu_i = \mu_i \), other \( \mu_k = \mu_k \).

**Remark:** The group of PL transformations generated by \( t_i \) is bigger than \( S_N^+ \) (in particular, \( t_i \) do not satisfy braid relations). It can be described as a quotient of the cactus group \( C_0 \).

Ref: Berenstein-Kirillov, Chmutov-Glick-Polyavsky

2. **Goal:** describe explicitly \( gl_n \otimes L(\lambda) \) in the basis \( \sum_{T \in GT(\lambda)} \).

**Idea:** Consider a bigger set of generators for \( U(gl_n) \) that includes all generators of \( ZU(gl) \) for \( k \in \mathbb{N} \) that are diagonal in the basis \( \{ \mathbf{e}_i \} \).

Describe the relations for this set of generators simultaneously; we'll solve the branching problem for the restriction \( \text{Res}_{gl_n}^{gl_{N-n}} V(\lambda) \) for any \( n \).

**Preliminary Answer:** \( L(\lambda) \otimes \bigoplus_{\lambda'} M_{\lambda'} L(\lambda') \)

\( M_{\lambda'} \) is the multiplicity space, simple module over \( U(gl_n) \). Basis in \( M_{\lambda'} \) is given by

Keystone \( GT^+ \), table

\( U(gl_n) \) has a similar set of generators w. a set of relations independent on \( N \).
Generators of $\mathfrak{u}(\mathfrak{gl}_N)_{GL_N}$.

First approximation: $S(\mathfrak{gl}_N) = \mathfrak{u}(\mathfrak{gl}_N)$ (PBW)

Moreover, the symmetrization map is a $\mathfrak{gl}_N$-module iso

$S(\mathfrak{gl}_N) \xrightarrow{\text{symmetrization}} \mathfrak{u}(\mathfrak{gl}_N)_{GL_N} \xrightarrow{T} \mathfrak{u}(\mathfrak{gl}_N)$

$\Rightarrow S(\mathfrak{gl}_N) = \mathfrak{u}(\mathfrak{gl}_N)_{GL_N}$, so can lift some generators of $S(\mathfrak{gl}_N)_{GL_N}$ to get generators of $\mathfrak{u}(\mathfrak{gl}_N)_{GL_N}$.

$S(\mathfrak{gl}_N) = \mathbb{C}[\mathbb{M}^N] = \mathbb{C} \left( \frac{V^* \otimes V}{V \otimes V^*} \right)_G$

by Fundamental Thm, all invariants come from pairings $V \otimes V^* \rightarrow \mathbb{C}$, so they are polynomials of

- $(V \otimes V^*) \otimes (V \otimes V^*)$ - trace of $r$-th power up to divisible by the next
- $V^* \otimes (V \otimes V^*)$ - $r$-th power of the whole thing up to divisible by the next
- matrix elements of the lower-right corner.

Equivalently, we have the following generators:

Thm: $S(\mathfrak{gl}_N)_{GL_N}$ is generated by $\text{Tr} E^{r \times (E)_{n_i \times n_j}}_{i,j \leq n, \quad r = 1, 2, 3, \ldots}$, $E = (E_{i,j}) \in S(\mathfrak{gl}_N) \otimes \text{Mat}_N$.

Cor: the same generators for $\mathfrak{u}(\mathfrak{gl}_N)_{GL_N}$.

Relations: Let $a_r = \text{Tr} E^r$, $t^{(r)}_{i,j} = \left( E^{r \times (E)_{n_i \times n_j}} ight)_{n_i \times n_j}$

Theorem: the following relations hold: 1) $a_r$ are central

2) $[t^{(r)}_{i,j}, t^{(s)}_{k,l}] = t^{(r)}_{i,j} t^{(s)}_{k,l} - t^{(s)}_{k,l} t^{(r)}_{i,j} \quad (\text{convention: } t^{(0)}_{i,j} = \delta_{i,j})$.

Def: $Y(\mathfrak{gl}_N)$ - Yangian of $\mathfrak{gl}_N$ - is the algebra generated by $t^{(r)}_{i,j}$ modulo the above relations ($\ast$).