Recitation 8
Tuesday October 5, 2021

1 Recap

1.1 Orthogonality of Vectors

Let \( u \) and \( v \) be vectors of the same dimension. We say \( u \) and \( v \) are orthogonal iff their angle is 90°, or equivalently \( u \cdot v = u^\top v = v^\top u = 0 \).

In addition, we say that a set of vectors \( \{v_1, v_2, ..., v_n\} \) is pairwise orthogonal iff \( v_i \) and \( v_j \) are orthogonal for any \( i \neq j \in \{1, 2, ..., n\} \). A set of pairwise orthogonal (nonzero) vectors is always linearly independent.

A set of vectors \( \{u_1, u_2, ..., u_n\} \) is pairwise orthonormal if it is pairwise orthogonal, and each \( u_i \) is a unit vector.

1.2 Orthogonality of Subspaces

Two subspaces \( U \) and \( V \) of \( \mathbb{R}^n \) are orthogonal if \( u \cdot v = 0 \) for all \( u \in U \) and \( v \in V \). In addition, it follows that \( \dim U + \dim V \leq n \).

1.3 Orthogonal Complement of Subspaces

Given a subspace \( V \), its orthogonal complement \( V^\perp \) is defined as:

\[
V^\perp = \{ w : w \cdot v = 0 \text{ for any } v \in V \}.
\]

Intuitively, \( V^\perp \) is the largest subspace that is orthogonal to \( V \).

Some important properties include

1. \( \dim V + \dim V^\perp = n \)

2. \( (V^\perp)^\perp = V \)

1.4 Decomposition

**Theorem 1** Let \( V, W \subseteq \mathbb{R}^n \) are orthogonal complements – that is \( V = W^\perp \) and \( W = V^\perp \).

Then every vector \( x \in \mathbb{R}^n \) has a unique decomposition \( x = v + w \) where \( v \in V \) and \( w \in W \). In addition, it follows that \( v \cdot w = 0 \).
1.5 Some Familiar Orthogonal Complements

We have already seen and worked on orthogonal complements, but we just didn’t realize that they are!

**Theorem 2** $N(A)$ and $C(A^T)$ are orthogonal complements in $\mathbb{R}^n$. Similarly, $C(A)$ and $N(A^T)$ are orthogonal complements in $\mathbb{R}^m$.

In relation to Theorem 1, we can plug in $V = N(A)$ and $W = C(A^T)$ and derive the following result.

**Theorem 3** Suppose that we are given a matrix $A \in \mathbb{R}^{m \times n}$. Any vector $v \in \mathbb{R}^n$ can be written uniquely as $v = v_1 + v_2$ where $v_1 \in N(A)$ and $v_2 \in C(A^T)$.

1.6 Relationship to Projection

Suppose that we want to project a vector $v$ onto a unit vector $w$, then the projection is

$$\text{proj}_w v = (v \cdot w) w.$$ 

We note that $v \cdot w$ is a scalar – which ensures that the projection is on $w$. In general cases where $w$ is not necessarily a unit vector, we have

$$\text{proj}_w v = \left( \frac{v \cdot w}{\|w\|^2} \right) w.$$ 

1.7 Gram-Schmidt

Let’s suppose that we a set $V = \{v_1, ..., v_k\}$ of linearly independent vectors. Our goal is to transform it into a set of orthonormal vectors $W$.

**Algorithm 1** GRAM-SCHMIDT

**Input:** a set $V = \{v_1, ..., v_k\}$ of linearly independent vectors

- $w_1 := \text{normalize}(v_1)$
- $w_2 := \text{normalize}(v_2 - \text{proj}_{w_1} v_2)$
- $w_3 := \text{normalize}(v_3 - \text{proj}_{w_1} v_3 - \text{proj}_{w_2} v_3)$
- ...
- $w_k := \text{normalize} \left( v_k - \sum_{i=1}^{k-1} \text{proj}_{w_i} v_k \right)$

**Output** $W = \{w_1, ..., w_k\}$

One crucial property is that $V$ and $W$ span the same subspace. In other words, if we are given a subspace $S$ which is the span of basis $V$, we can use Gram-Schmidt to derive its orthonormal basis $W$ – meaning that $W$ is a basis of $S$ and is orthonormal.
2 Exercises

1. Among the following six 3-dimensional vectors, which pairs are orthogonal?

\[
\begin{align*}
  a &= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \\
  b &= \begin{bmatrix} 2 \\ -6 \\ -3 \end{bmatrix}, \\
  c &= \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \\
  d &= \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \\
  e &= \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}, \\
  f &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{align*}
\]

2. Denote a subspace \( V = \{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} : 2v_1 + 3v_3 + 5v_5 = 0 \} \). Find \( V^\perp \).

3. Suppose that we have a subspace \( S \) with an orthogonal basis \( \{ v_1, ..., v_k \} \). By the definition of basis, any vector \( v \in S \) can be expressed as

\[
v = \sum_{i=1}^{k} \alpha_i v_i = \alpha_1 v_1 + ... \alpha_k v_k
\]

for some constants \( \alpha_1, ..., \alpha_k \). Determine each \( \alpha_j \) in terms of \( v_1, ..., v_k \) and \( v \). Will the same derivation work if it not for the orthogonality of \( \{ v_1, ..., v_k \} \)?

4. Suppose that a set of vectors \( \{ v_1, ..., v_n \} \) generates a subspace \( S \). In other words, \( S = \text{Span}\{ v_1, ..., v_n \} \). Describe a procedure to derive an orthonormal basis of \( S \).

5. In this problem, we will explore the effect of ordering on the Gram-Schmidt algorithm. Denote

\[
\begin{align*}
  u_1 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\
  u_2 &= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \\
  u_3 &= \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}
\end{align*}
\]

and

\[
\begin{align*}
  v_1 &= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \\
  v_2 &= \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}, \\
  v_3 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{align*}
\]

for which each \( \{ u_1, u_2, u_3 \} \) and \( \{ v_1, v_2, v_3 \} \) is a set of three linearly independent vectors. Moreover, the two sets \( \{ u_1, u_2, u_3 \} \) and \( \{ v_1, v_2, v_3 \} \) are identical, but are in different orders. This means both sets are bases of the same subspace \( S \).

(a) Perform Gram-Schmidt on \( \{ u_1, u_2, u_3 \} \) to derive an orthonormal basis of \( S \).

(b) Perform Gram-Schmidt on \( \{ v_1, v_2, v_3 \} \) to derive an orthonormal basis of \( S \).

(c) Each of the answer to the previous parts is an orthonormal basis of \( S \). Are they identical? What can we conclude about the effect of order to the Gram-Schmidt?