Last time: V finite dim./k, B non-degenerate symmetric bilinear form $\Rightarrow$
$\exists$ orthogonal basis $(e_1, \dots, e_n)$, $B(e_i, e_j) = 0 \ \forall i \neq j$
+ over $\mathbb{C}$, can ensure $B(e_i, e_i) = 1$
$\mathbb{R}$ $\qquad B(e_i, e_i) = \pm 1$

* What about the skew-symmetric case? (suppose $\mathrm{char}(k) \neq 2$)

We can still find a "standard basis" for V finite dim. vect. space with $B: V \times V \to k$ nondegenerate skew-symmetric bilinear form (a.k.a: symplectic form) but the process is slightly different since $B(v,v) = 0 \ \forall v \in V$.

Instead: pick any nonzero $e_1 \in V$; since B is nondegenerate, $B(e_1, \cdot) : V \to k$ is nonzero $\Rightarrow \exists f_1 \in V$ st. $B(e_1, f_1) \neq 0$, can make it $= 1$ by scaling f_1.

Now we find $\mathrm{span}(e_1, f_1) \cap \mathrm{span}(e_1, f_1)^\perp = \{0\}$ (if $v = ae_1 + bf_1$ has $B(v, e_1) = B(v, f_1) = 0 \Rightarrow a = b = 0$)
so $V = \mathrm{span}(e_1, f_1) \oplus \mathrm{span}(e_1, f_1)^\perp$,
and study the restriction of B to the latter subspace (induction on dim.).

$\Rightarrow$ Prop: V finite dim.^{l} over k, $\mathrm{char}(k) \neq 2$,
B nondegenerate skewsymmetric bilinear form $V \times V \to k$
$\Rightarrow \dim V$ is even, and V has a basis $(e_1, f_1, \dots, e_n, f_n)$ st.
$B(e_i, e_j) = B(f_i, f_j) = 0$, $B(e_i, f_j) = \delta_{ij} = -B(f_j, e_i)$.

ie. matrix of B is
$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \\ & & \ddots \end{pmatrix}$$

The group of linear transformations preserving B is called the symplectic group $Sp(V, B) \simeq Sp(2n, k)$.

Tensors and multilinear algebra — see handout.

V, W finite dimensional vector spaces over k $\Rightarrow$ the tensor product is a vector space $V \otimes W$ + a bilinear map
$$V \times W \to V \otimes W$$
$$(v, w) \mapsto v \otimes w$$

Three definitions (from concrete to abstract; all are equivalent ie. give same output up to natural isomorphism)

- <u>Def. 1:</u> Choose bases $e_1 \dots e_m$ of V, $f_1 \dots f_n$ of W. Then $V \otimes W$ is the vector space with basis $\{e_i \otimes f_j,\ 1 \leq i \leq m,\ 1 \leq j \leq n\}$.

  The bilinear map is $(e_i, f_j) \mapsto e_i \otimes f_j$ + extend by linearity.

  Elements of the form $v \otimes w = (\sum a_i e_i) \otimes (\sum b_j f_j) = \sum a_i b_j \,(e_i \otimes f_j)$ are called <u>pure tensors</u>; not every element of $V \otimes W$ is of this form!
  The <u>rank</u> of an element of $V \otimes W$ = minimal number of terms needed to express it as a linear combination of pure tensors.

This is concrete & makes it clear that $\dim(V \otimes W) = mn$, but the independence of the choice of basis isn't obvious. To de-emphasize the basis:

- <u>Def 2:</u> Start with a vector space U with basis $\{v \otimes w \,/\, v \in V, w \in W\}$. (This is insanely large: usually this basis is uncountable!), and quotient it by a subspace R of relations among these elements:

$$\begin{aligned} R \subset U = \text{the span of } & (\lambda v) \otimes w - \lambda (v \otimes w) \quad \forall \lambda, v, w \\ & v \otimes (\lambda w) - \lambda (v \otimes w) \\ & (u+v) \otimes w - u \otimes w - v \otimes w \quad \forall u, v, w. \\ & u \otimes (v+w) - u \otimes v - u \otimes w \end{aligned}$$

  Defining $V \otimes W = U/R$ sets all these to zero, enforcing bilinearity of the map $(v,w) \mapsto v \otimes w$.

This shows independence on the basis, but involves an unpleasantly large construction (at the end, if we have bases $\{e_i\}$ of V, $\{f_j\}$ of W, the relations in R do show all elements of $V \otimes W$ are linear combinations of $e_i \otimes f_j$, but before one checks this it's not even obvious that $\dim(V \otimes W) < \infty$)

- The least concrete, yet most mathematically satisfactory definition, characterizes what $V \otimes W$ does without spelling out how it's actually constructed: namely, that $V \otimes W$ is the largest space we can build s.t. a linear map from $V \otimes W$ to another space, when evaluated on pure tensors $v \otimes w$, is bilinear in v and w. (eg. in Def. 2: U is too big, quotient by R enforces bilinearity)

<u>Def 3.:</u> The tensor product $V \otimes W$ is the universal vector space through which all bilinear maps from $V \times W$ factor, ie. it is a vector space $V \otimes W$ + a bilinear map $\beta: V \times W \to V \otimes W$ such that, given any vector space U over k, and any bilinear map $b: V \times W \to U$, there exists a unique linear map $\varphi: V \otimes W \to U$ st. $b = \varphi \circ \beta$

$$\begin{tikzcd} V\times W \arrow[r, "b"] \arrow[d, "\beta"'] & U \\ V\otimes W \arrow[ur, dashed, "\exists!\varphi"'] & \end{tikzcd}$$

This tells us the key property of $V\otimes W$ and implies uniqueness up to isomorphism (the univ. property gives isom's between any two candidate constructions of $V\otimes W$), but existence ultimately comes from one of the previous constructions!

<u>Check</u>: Def. 1 satisfies the property: given bases $\{e_i\}$ & $\{f_j\}$ of U and V

$$\{\text{bilinear maps } b: V\times W \to U\} \longleftrightarrow \{\text{linear maps } \varphi: V\otimes W \to U\}$$

by defining $b(e_i, f_j) = \varphi(e_i\otimes f_j)$ and vice versa.

<u>Basic properties</u>:

- $\otimes : Vect_k \times Vect_k \to Vect_k$ is a functor. This means: given linear maps $\left\{\begin{array}{l} f: V\to V' \\ g: W\to W' \end{array}\right\}$ we get a linear map $f\otimes g: V\otimes W \to V'\otimes W'$ on pure elements: $(f\otimes g)(v\otimes w) = f(v)\otimes g(w)$. and this respects composition.
- $V\otimes W \cong W\otimes V$ (natural isom., could even claim they're equal...)
- $(U\oplus V)\otimes W \cong (U\otimes W)\oplus(V\otimes W)$

More surprising but extremely useful: $\boxed{Hom(V,W) \simeq V^*\otimes W}$

<u>Proof</u>: the map
$$V^*\times W \longrightarrow Hom(V,W)$$
$$(\ell, w) \longmapsto (v\mapsto \ell(v)w)$$
is bilinear

so by univ. property we get a linear map $V^*\otimes W \longrightarrow Hom(V,W)$ which takes $\ell\otimes w \mapsto (v\mapsto \ell(v)w)$.

Pick bases $(e_1 \dots e_n)$ of V, $(f_1 \dots f_m)$ of W, let $(e_1^* \dots e_n^*)$ dual basis of V^*.

Then $(e_i^*\otimes f_j)$ basis of $V^*\otimes W$.

The above construction takes $(e_i^*\otimes f_j)$ to $\varphi_{ij}: \begin{array}{l} V\longrightarrow W \\ v\longmapsto e_i^*(v)\, f_j \end{array}$

whose action on basis vectors is: e_i maps to f_j, all others to 0.

Thus $\mathcal{M}(\varphi_{ij}) =$ $m\times n$ matrix with a single nonzero entry $\begin{array}{c c} & i \\ j & \begin{pmatrix} & \vdots & \\ \cdots & 1 & \\ & & \end{pmatrix} \end{array}$

These form a basis of $Hom(V,W)$.

Since it maps a basis to a basis, $V^*\otimes W \to Hom(V,W)$ is an isom. $\square$

* <u>Ex</u>: if V has basis (e_1, e_2), V^* (e_1^*, e_2^*), & W has basis f_1, f_2, then the linear map with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $e_1^*\otimes(af_1+cf_2) + e_2^*\otimes(bf_1+df_2)$.

This is in general a rank 2 tensor, except if $ad-bc=0$, then can write it as a pure tensor $(xe_1^*+ye_2^*)\otimes(zf_1+wf_2)$

<u>Fact:</u> Tensor rank in $V^*\otimes W$ is the same as rank in $\mathrm{Hom}(V,W)$! (hence the name).

(Rank 1 case: $\ell\otimes w$ corresponds to $(v\mapsto \ell(v)w)$ whose image $=\mathrm{span}(w)$!)

Easiest to see if take basis of V in which $e_{r+1}\dots e_n$ basis of $\ker\varphi$ and of W in which $f_1\dots f_r$ basis of $\mathrm{Im}\,\varphi$, with $f_i=\varphi(e_i)\ \forall 1\le i\le r$.

Then φ corresponds to $\sum_{i=1}^r e_i^*\otimes f_i$. $\left(\Leftrightarrow \mathcal{M}(\varphi)=\left(\begin{array}{c|c} \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & 0 \\ \hline 0 & 0\end{array}\right)\right)$ $\underset{r=\operatorname{rank}\varphi}{\longleftrightarrow}$

The isomorphism $\mathrm{Hom}(V,W)\simeq V^*\otimes W$ also implies:

- $(V\otimes W)^*\simeq V^*\otimes W^*$. Can view this as:

$$(V\otimes W)^* = \mathrm{Hom}(V\otimes W,k) = \{\text{Bilinear maps } V\times W\to k\}$$
$$\simeq \mathrm{Hom}(V,W^*) \quad (\text{via } b\mapsto \varphi_b: \underset{\in V}{v}\mapsto \underset{\in W^*}{b(v,\cdot)})$$
$$\simeq V^*\otimes W^*$$

- $\mathrm{Hom}(V,W)\simeq V^*\otimes W\simeq (W^*)^*\otimes V^*\simeq \mathrm{Hom}(W^*,V^*)$

This is actually the <u>transpose</u> construction $\varphi\in\mathrm{Hom}(V,W)\mapsto \varphi^t: W^*\to V^*$.

(easiest to check on rank 1 $\varphi(v)=\ell(v)w \rightsquigarrow \varphi^t(\alpha)=\alpha\circ\varphi=\alpha(w)\,\ell=ev_w(\alpha)\ell$
$\ell\otimes w\rightsquigarrow ev_w\otimes \ell$.)

- We can now properly define the <u>trace of a linear operator</u> !

In "ordinary" linear algebra classes, one defines the trace of an $n\times n$ matrix $A=(a_{ij})$ to be $tr(A)=\sum_{i=1}^n a_{ii}$ sum of diagonal entries, then noting that $tr(AB)=\sum_{i,j}a_{ij}b_{ji}=tr(BA)$ we have $tr(P^{-1}AP)=tr(A)$ and so the trace of $T:V\to V$ is defined to be the trace of $\mathcal{M}(T)$ in any basis.

We could also try to define the trace via eigenvalues and their multiplicities, over an alg. closed field: in a basis where $\mathcal{M}(T)$ is triangular it is manifest that $tr(T)=\sum n_i\lambda_i$

- We can do better (conceptually), by using $\mathrm{Hom}(V,V)\simeq V^*\otimes V$, and the <u>contraction</u> linear map $V^*\otimes V\to k$. Namely, there's a natural bilinear pairing $ev : V^* \times V \to k$, $(\ell, v) \mapsto \ell(v)$ and it determines $tr : V^\vee \otimes V \to k$ on pure tensors, $\ell \otimes v \mapsto \ell(v)$

This is indeed equivalent to the usual def^n: choosing a basis (e_i) and the dual basis (e_i^*), $tr(e_i^\vee \otimes e_j) = e_i^*(e_j) = \delta_{ij}$ $\longleftrightarrow$ trace of the matrix with single entry 1 in pos. (j,i).

Def. A map $m : V_1 \times \dots \times V_k \to W$ is <u>multilinear</u> if it is linear in each variable separately.

The tensor product $V_1 \otimes \dots \otimes V_k$ can be defined as above, either using bases of $V_1 \dots V_k$, or as a quotient of a universal vector space by relations, or via universal property for multilinear maps:

There is a multilinear map $\mu : V_1 \times \dots \times V_k \to V_1 \otimes \dots \otimes V_k$ s.t.
$(v_1, \dots, v_k) \mapsto v_1 \otimes \dots \otimes v_k$

$\forall W$ vector space, $\forall\, m : V_1 \times \dots \times V_k \to W$ multilinear, $\exists!\, \varphi \in Hom(V_1 \otimes \dots \otimes V_k, W)$ s.t. $m = \varphi \circ \mu$

$$\begin{tikzcd} V_1 \times \dots \times V_k \arrow[r, "m"] \arrow[d, "\mu"'] & W \\ V_1 \otimes \dots \otimes V_k \arrow[ur, dotted, "\exists!\varphi"'] & \end{tikzcd}$$

In fact nothing new is happening, because $(U \otimes V) \otimes W = U \otimes (V \otimes W) = U \otimes V \otimes W$.

But ... in the special case of $\underbrace{V \otimes \dots \otimes V}_{n \text{ times}} = V^{\otimes n}$ (by convention $V^{\otimes 0} = k$, $V^{\otimes 1} = V$)

we have bilinear maps $V^{\otimes k} \times V^{\otimes \ell} \to V^{\otimes (k+\ell)}$ $\forall k, \ell \geq 0$, which taken together define a multiplication on the <u>tensor algebra</u> $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ making it a noncommutative ring.