Math 55a Homework 11

Due Wednesday November 18, 2020.

Material covered: Representations; Schur's lemma; representations of S_3. (Fulton-Harris §1)

1. Show that, for any finite abelian group G, there is a natural map from G to its double dual $\widehat{\widehat{G}}$, and that this map is an isomorphism. (Recall the dual of G is $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$).

2. Let G be a group and V and W representations of G, and let $\mathrm{Hom}(V, W) = V^* \otimes W$ be given the structure of a representation as seen in class: g maps $\varphi \in \mathrm{Hom}(V, W)$ to $g \circ \varphi \circ g^{-1}$ (where g is acting on W and g^{-1} on V).

(a) Show that the invariant subspace $\mathrm{Hom}(V, W)^G = \{\varphi \in \mathrm{Hom}(V, W) \,|\, g\varphi = \varphi \ \forall g \in G\}$ is the vector space of homomorphisms $\varphi : V \to W$ of representations, that is, G-equivariant linear maps. (This is often denoted $\mathrm{Hom}_G(V, W)$).

(b) Suppose that the irreducible representations of G are $U_1, U_2, \ldots, U_c$, with $\dim(U_i) = d_i$, and suppose that $V = U_1^{\oplus m_1} \oplus \cdots \oplus U_c^{\oplus m_c}$ and $W = U_1^{\oplus n_1} \oplus \cdots \oplus U_c^{\oplus n_c}$. What is the dimension of $\mathrm{Hom}_G(V, W)$?

3. Let U, V and W be vector spaces (not necessarily finite-dimensional).

(a) Construct a canonical isomorphism $\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W))$.

(b) Now suppose that U, V and W are representations of a group G. Show that the isomorphism of part (a) is in fact an isomorphism of representations.

4. Let V be any (finite-dimensional) representation of a group G.

(a) Show that V is irreducible if and only if V^* is.

(b) If W is any 1-dimensional representation, show that V is irreducible if and only if $V \otimes W$ is.

5. Let V be the standard (2-dimensional) representation of S_3.

(a) Identify $\mathrm{Sym}^4 V$ as a direct sum of irreducible representations of S_3.

(b) Identify $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ as a direct sum of irreducible representations of S_3.

(c) On a previous homework (HW7), you constructed for any 2-dimensional vector space V a natural map $\phi : \mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$. Show that this is a homomorphism of representations and identify the kernel of ϕ as a representation of S_3.

6. Again, let V be the standard representation of S_3. Show that $\mathrm{Sym}^2(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^3(\mathrm{Sym}^2 V)$ as representations of S_3.

7. Show by example that, over fields of characteristic $p > 0$, complete reducibility may fail. In other words, find an example of a finite group G, a finite-dimensional vector space over $\mathbb{F}_p$, an action of G on V (that is, a homomorphism $\rho : G \to GL(V)$) and a subspace $W \subset V$ invariant under G such that no complementary invariant subspace of V exists.

8. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?