Given a poset $P$, a (closed) interval $[s,t]$ is the induced subposet
\[ \{ u \mid u \geq s \text{ and } u \leq t \} \]

A poset is said to be \underline{locally finite} if all its intervals are finite.

E.g. $(\mathbb{N}, \leq)$ is an infinite, but locally finite poset.

All finite posets are locally finite.

**Definition**

Let $P$ be a locally finite poset, and $\text{Int}(P)$ the set of its intervals.

The incidence algebra of $P$, $\mathcal{I}(P)$ is the set of all functions $f: \text{Int}(P) \to \mathbb{R}$ under the operations of

- \textbf{addition}

  \[(f+g)(x,z) = f(x,z) + g(x,z)\]

  \text{for interval } [x,z]

- \textbf{scalar multiplication}, with $c \in \mathbb{R}$

  \[(c \cdot f)(x,z) = c \cdot f(x,z)\]

- \textbf{convolution product}

  \[(f \ast g)(x,z) = \sum_{y \leq z} f(x,y) g(y,z)\]

**Theorem**

If $P$ is a locally finite poset, $\mathcal{I}(P)$ is an associative algebra.

The proof is straightforward and not very enlightening. However, we should point out what the identity element is for the convolution:

\[\delta(x,z) = \begin{cases} 1 & \text{if } x = z \\ 0 & \text{otherwise.} \end{cases}\]
Proposition

Let $f \in I(P)$. The following conditions are equivalent:

(i) $f$ has a left inverse
(ii) $f$ has a right inverse
(iii) $f$ has a two-sided inverse
(iv) $f(\mathbf{t}, \mathbf{t}) \neq 0 \forall \mathbf{t} \in P$.

Moreover, if $f^{-1}$ exists, then $f^{-1}(s, t)$ depends only on $(s, t)$.

Some elements of the incidence algebra

- The zeta function $\zeta$, defined by
  \[ \zeta(t, u) = 1 \text{ for all } t \leq u \text{ in } P. \]

- $\zeta^2$:
  \[ \zeta^2(s, u) = \sum_{s \leq t \leq u} 1 = \#(s, u). \]

- Or more generally:
  \[ \zeta^k(s, u) = \sum_{s = s_0 \leq s_1 \leq \ldots \leq s_k \leq u} 1 \]
  is the number of multichains (i.e., chains with repeated elements) from $s$ to $u$.

- Similarly,
  \[ (\zeta^{-1})(s, u) = \begin{cases} 1 & \text{if } u = s \\ 0 & \text{otherwise} \end{cases} \]

- And
  \[ (\zeta^{-1})^k(s, u) \]
  is the number of (regular) chains from $s$ to $u$.

- Consider $Z - \zeta = 2\zeta - \zeta$:
  \[ (Z - \zeta) = \begin{cases} 1 & \text{if } s = t \\ -1 & \text{if } s < t \end{cases} \]

By the proposition from last page, $Z - \zeta$ is invertible.
Claim: \((2-\delta)^d\) is the total number of chains.

Sketch of proof: Let \(P\) be the length of the longest chain in the interval \([s,u]\). Then, \((2-\delta)^{d+1}(b,v) = 0\) for all \(s \leq b \leq v \leq u\). Thus, we have

\[
(2-\delta)^{d+1}(b,v) = \left[ 1 - (2-\delta)^m \right] \left[ 1 + (2-\delta)^1 + (2-\delta)^2 + \cdots + (2-\delta)^m \right] (b,v)
\]

Hence, \((2-\delta)^d = 1 + (2-\delta)^1 + \cdots + (2-\delta)^d\) is the total number of chains in the interval.

- \(\zeta\) function,

\[
\zeta(s,t) = \begin{cases} 1 & \text{if } t \text{ covers } s \\ 0 & \text{otherwise} \end{cases}
\]

- and

\[(1-\zeta)(s,t)
\]

is the number of maximal chains in the interval \([s,t]\).

Proof: exercise.

- Möbius function

The inverse of the \(\zeta\) function is the Möbius function, defined as

\[
\mu(s,t) = \begin{cases} 1 & \text{if } s \leq t \\ -\sum_{s \leq m \leq t} \mu(s,m), \text{ otherwise} \end{cases}
\]

If a poset has a \(\hat{0}\), we also write \(\mu(X) = \mu(\hat{0},X)\).

To be able to prove it, consider the following lemma.

Lemma

\[
\sum_{x \leq y \leq z} \mu(y,z) = \sum_{x \leq y \leq z} \mu(x,y).
\]
Proof
By definition of $\mu$, if $x=2$, then $\sum \mu(x,y) = \mu(2,1) = 1$.

Otherwise, $\sum_{x \leq y \leq 2} \mu(x,y) = \sum_{x \leq y \leq 2} \mu(x,y) + \mu(y,2) = \sum_{x \leq y \leq 2} \mu(x,y) - \sum_{x \leq y \leq 2} \mu(x,y(y,2)) = 0$.

Proof of the inverse
$(\mu * \delta)(x,2) = \sum_{x \leq y \leq 2} \mu(x,y) \delta(y,2) = \sum_{x \leq y \leq 2} \mu(x,y) \delta(y,2) = \sum_{x \leq y \leq 2} \mu(x,y) = \delta(x,2)$

Example
- Compute $\mu(x)$ for $x \in [n] = \{1, 2, \ldots, n\}$.

$\mu(x) = \begin{cases} 1 & \text{if } x = 1 \\ -1 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$

Proposition
The Möbius function of any join irreducible element $x$ in a lattice is $\mu(x) = 0$ if $x$ is not covering $\hat{0}$.

Proof
$x$ is join irreducible iff it covers exactly one element $y$.

Then,

$\mu(x) = \sum_{z \leq x} -\mu(z) = \sum_{z \leq y} -\mu(z) - \sum_{z \geq y} -\mu(z) = 0$

- Compute $\mu(x)$ for $x \in 2^{[n]}$ (the boolean lattice).

$\mu(x) = (-1)^{#x}$ (here, $x$ is a subset of $2^{[n]}$)

Proof
Recall that $\sum_{x \leq y} \mu(x) = \delta_{\hat{0}, x}$. This uniquely defines $\mu$, so we only need to check that $\sum_{y \leq x} (-1)^{#y} = 0$ if $x \neq \emptyset$.

$\sum_{y \leq x} (-1)^{#y} = \sum_{y \leq x} (-1)^{#x} * 1^{(#x)} = 0$
Compute \( \mu(n) \) for the divisors lattice.

(example: \( D_{18} \), can we see a pattern?)

\[
\mu(n) = \begin{cases} 
(-1)^m & \text{if } n \text{ is the product of } m \text{ distinct primes} \\
0 & \text{otherwise.}
\end{cases}
\]

We prove only a part of it.

If \( n \) is the product of \( m \) distinct primes, we prove it by recurrence.

- If \( n = 1 \), then \( \mu(1) = 1 \).
- Otherwise, \( \mu(n) = \sum_{d \mid n} \mu(d) \frac{1}{a(n)} \leq \sum_{k \leq m} (\mu_k(a))^k \leq \sum_{k \leq m} (\mu_k(a))^k + (\mu_k(a)^{-1})^k = (-1)^m \)

where \( \mu \) is because every number is a product of \( k \) different primes, and by induction hypothesis.

Also, if \( n \) is a power of a prime number, let's say \( n = p^k, k > 2 \), then \( p^k \) covers exactly one element and \( \mu(p^k) \) thus vanishes.

The Möbius function appears a lot in number theory, where it has exactly the definition above. The Prime Number's Theorem can be restated as \( \sum_{n=1}^\infty \frac{\mu(n)}{n} = 0 \) (This is not obvious).
Proof of (a) ( = 5) (everything else is similar).

Assume \( f(x) = \sum_{y \in X} g(y) \) for all \( x \in \mathcal{P} \). Then,

\[
\sum_{y \in X} \mu(x,y) f(y) = \sum_{y \in X} \mu(x,y) \sum_{z \in Y} g(z) = \sum_{z \in X} g(z) \sum_{x \in Y \subseteq z} \mu(x,y)
\]

\[
= \sum_{z \in X} g(z) \mathcal{S}_{X,Z}
\]

\[
= g(x).
\]

Application

**Principle of Inclusion and Exclusion. (PIE)**

Recall that, for the boolean lattice,

\[
\mu(S) = \begin{cases} 
1 & \text{if } \#S \text{ is even} \\
-1 & \text{if } \#S \text{ is odd} 
\end{cases}
\]

**Theorem (PIE)**

\[
|S - \bigcup_{i=1}^{n} S_i| = |S| - \sum_{i=1}^{n} |S_i| + \sum_{1 \leq i < j \leq n} |S_i \cap S_j| - \cdots + (-1)^{n} \prod_{i=1}^{n} |S_i|.
\]

The proof is a little bit involved, but it works with

\[
f(I) = \bigcap_{i \in I} S_i \quad \text{and} \quad g(I) = \bigcup_{i \in I} S_i - \bigcup_{j \in I} S_j = \text{elements in all } S_i, i \in I,
\]

\[
g(I) = \sum_{S \subseteq I, |I(S)| \neq |I|} f(S), \quad \text{and } I = \emptyset. \text{ It is also Theorem 5.5.7 in [AOJC].}
\]

References: [AOJC] Bruce E. Sagan. *Combinatorics, the art of counting*, §5.4, §5.5, §5.8.