1. (O 4.17) Let \( f : \{\text{True}, \text{False}\}^n \to \{\text{True}, \text{False}\} \) be computable by a CNF \( C \) of width \( w \). In this exercise you will show that \( I[f] \leq w \). Consider the following randomized algorithm that tries to produce an input \( x \in f^{-1}(\text{True}) \). First, choose a random permutation \( \pi \in S_n \). Then for \( 1 = 1, \ldots, n \): If the single-literal clause \( x_{\pi(i)} \) appears in \( C \), then set \( x_{\pi(i)} = \text{True} \), syntactically simplify \( C \) under this setting, and say that coordinate \( \pi(i) \) is “forced”. Similarly, if the single-literal clause \( \overline{x}_{\pi(i)} \) appears in \( C \), then set \( x_{\pi(i)} = \text{False} \), syntactically simplify \( C \), and say that \( \pi(i) \) is “forced”. If neither holds, set \( x_{\pi(i)} \) uniformly at random. If \( C \) ever contains two single-literal clauses \( x_j \) and \( \overline{x}_j \), the algorithm “gives up” and outputs \( x = \bot \).

(a) Show that if \( x \neq \bot \), then \( f(x) = \text{True} \).

Solution:

(b) For \( x \in f^{-1}(\text{True}) \) let \( p(x) = \Pr[x = x] \). For \( j \in [n] \) let \( I_j \) be the indicator random variable for the event that coordinate \( j \in [n] \) is forced. Show that \( p(x) = \mathbb{E}\left[\prod_{j=1}^{n}(1/2)^{1-I_j}\right] \).

Solution:

(c) Deduce \( 2^n p(x) \geq 2 \sum_{j=1}^{n} \mathbb{E}[I_j] \).

Solution:

(d) Show that for every \( x \) with \( f(x) = \text{True} \), \( f(x \oplus j) = \text{False} \) it holds that \( \mathbb{E}[I_j \mid x = x] \geq 1/w \).

Solution:

(e) Deduce \( I[f] \leq w \).

Solution:

2. (O 4.19) In this exercise you will prove the Baby Switching Lemma with constant 3 in place of 5. Let \( \phi = T_1 \lor T_2 \lor \cdots \lor T_s \) be a DNF of width \( w \geq 1 \) over variables \( x_1, \ldots, x_n \). We may assume \( \delta \leq 1/3 \), else the theorem is trivial.

(a) Suppose \( R = (J \mid z) \) is a “bad” restriction, meaning that \( \phi_{J\mid z} \) is not a constant function. Let \( i \) be minimal such that \( \phi_{J\mid z} \) is neither constantly True or False, and let \( j \) be minimal such that \( x_j \) or \( \overline{x}_j \) appears in this restricted term. Show there is a unique restriction \( R' = (J \setminus \{j\} \mid z') \) extending \( R \) that doesn’t falsify \( T_i \).

Solution:

(b) Suppose we enumerate all bad restrictions \( R \), and for each we write the associated \( R' \) as in (a). Show that no restriction is written more than \( w \) times.
Solution:

(c) If \( (J|z) \) is a \( \delta \)-random restriction and \( R \) and \( R' \) are as in (a), show that
\[
\Pr[(J|z) = R] = \frac{2\delta}{1-\delta} \Pr[(J|z) = R']
\]

Solution:

(d) Complete the proof by showing \( \Pr[(J|z) \text{ is bad}] \leq 3\delta w \)

Solution:

Lecture 9

1. (O 5.2) Let \( f(x) = \text{sgn}(a_0 + a_1 x_1 + \cdots + a_n x_n) \) be an LTF.
   (a) Show that if \( a_0 = 0 \), then \( \mathbb{E}[f] = 0 \). (Hint: Show that \( f \) is in fact an odd function.)

   Solution:

   (b) Show that if \( a_0 \geq 0 \), then \( \mathbb{E}[f] \geq 0 \). Show that the converse need not hold.

   Solution:

   (c) Suppose \( g : \{-1,1\}^n \to \{-1,1\} \) is an LTF with \( \mathbb{E}[g] = 0 \). Show that \( g \) can be represented as \( g(x) = \text{sgn}(c_1 x_1 + \cdots + c_n x_n) \)

   Solution:

2. (O 5.5) Suppose \( \ell : \{-1,1\}^n \to \mathbb{R} \) is defined by \( \ell(x) = a_0 + a_1 x_1 + \cdots + a_n x_n \). Define \( \hat{\ell} : \{-1,1\}^{n+1} \to \mathbb{R} \) is defined by \( \hat{\ell}(x) = a_0 x_0 + a_1 x_1 + \cdots + a_n x_n \). Show that \( \|\ell\|_1 = \|\hat{\ell}\|_1 \) and \( \|\ell\|_2^2 = \|\hat{\ell}\|_2^2 \).

   Solution:

3. (O 5.7) Consider the following “correlation distillation” problem (cf. Exercise 2.56). For each \( i \in [n] \) there is a number \( \rho_i \in [-1,1] \) and an independent sequence of pairs of \( \rho_i \)-correlated bits, \( (a_i^{(1)}, b_i^{(1)}), (a_i^{(2)}, b_i^{(2)}), (a_i^{(3)}, b_i^{(3)}), \ldots \) etc. Party A on Earth has access to the stream of bits \( a_i^{(1)}, a_i^{(2)}, a_i^{(3)}, \ldots \) and a party B on Venus has access to the stream \( b_i^{(1)}, b_i^{(2)}, b_i^{(3)}, \ldots \). Neither party knows the numbers \( \rho_1, \ldots, \rho_n \). The goal is for B to estimate these correlations. To assist in this, A can send a small number of bits to B. A reasonable strategy is for A to send \( f(a_i^{(1)}), f(a_i^{(2)}), f(a_i^{(3)}) \) to B, where \( f : \{-1,1\}^n \to \{-1,1\} \) is some Boolean function. Using this information B can try to estimate \( \mathbb{E}[f(a) b_i] \) for each \( i \).
(a) Show that $E[f(a)b_i] = \hat{f}(i)\rho_i$.

Solution:

(b) This motivates choosing an $f$ for which all $\hat{f}(i)$ are large. If we also insist all $\hat{f}(i)$ be equal, show that majority functions $f$ maximize this common value.

Solution:

4. (5.8) For $n \geq 2$, let $f : \{-1,1\}^n \to \{-1,1\}$ be a randomly chosen function (as in Exercise 1.7). Show that $\|f\|_\infty \leq 2\sqrt{n}/2$ except with probability at most $2^{-n}$.

Hint: First, consider one Fourier coefficient. Then apply a union bound.

Solution: