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II-2: Scalar QED

1 Introduction

Now that we have Feynman rules and we know how to quantize the photon, we are very close to quantum electrodynamics. All we need is the electron, which is a spinor. Before we get into spinors, however, it is useful to explore a theory which is an approximation to QED in which the spin of the electron can be neglected. This is called scalar QED. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + |D_\mu\phi|^2 - m^2|\phi|^2 \tag{1}$$

with

$$\begin{aligned} D_\mu\phi &= \partial_\mu\phi + ieA_\mu\phi \\ D_\mu\phi^\star &= \partial_\mu\phi^\star - ieA_\mu\phi^\star \end{aligned} \tag{2}$$

Although there actually do exist charged scalar fields in nature which this Lagrangian describes, for example the charged pions, that's not the reason we are introducing scalar QED before spinor QED. Spinors are somewhat complicated, so starting with this simplified Lagrangian will let us understand some elements of quantum electrodynamics without having to deal with spinor algebra.

2 Quantizing complex scalar fields

We saw that for a scalar field to couple to A_μ it has to be complex. This is because the charge is associated with a continuous global symmetry under which

$$\phi \to e^{-i\alpha}\phi \tag{3}$$

Such phase rotations only make sense for complex fields. The first thing to notice is that the classical equations of motion for ϕ and $\phi^\star$ are[1]

$$(\Box + m^2)\phi = i(-eA_\mu)\partial_\mu\phi + i\partial_\mu(-eA_\mu\phi) + (-eA_\mu)^2\phi \tag{4}$$

$$(\Box + m^2)\phi^\star = i(eA_\mu)\partial_\mu\phi^\star + i\partial_\mu(eA_\mu\phi^\star) + (eA_\mu)^2\phi^\star \tag{5}$$

So we see that ϕ and $\phi^\star$ couple to the electromagnetic field with opposite charge, but have the same mass. Of course, something having an equation doesn't mean we can produce it. However, in a second-quantized relativistic theory, the radiation process, $\phi \to \phi\gamma$, automatically implies that $\gamma \to \phi\phi^\star$ is also possible (as we will see). Thus, we must be able to produce these $\phi^\star$ particles. In other words, in a relativistic theory with a massless spin 1 field, **antiparticles** must exist and we know how to produce them!

To see antiparticles in the quantum theory, first recall that a quantized real scalar field is

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-ipx} + a_p^\dagger e^{ipx}\right) \tag{6}$$

Since a complex scalar field must be different from its conjugate by definition, we have to allow for a more general form. We can do this by introducing two sets of creation an annihilation operators and writing

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-ipx} + b_p^\dagger e^{ipx}\right) \tag{7}$$

1. We are treating ϕ and $\phi^\star$ as separate real degrees of freedom. If you find this confusing you can always write $\phi = \phi_1 + i\phi_2$ and study the physics of the two independent fields ϕ_1 and ϕ_2, but the ϕ and $\phi^\star$ notation is much more efficient.

Then, by complex conjugation

$$\phi^\star(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p^\dagger e^{ipx} + b_p e^{-ipx} \right) \tag{8}$$

Thus we can conclude that b_p annihilates particles of the opposite charge and same mass to what a_p annihilates. That is, b_p annihilates the antiparticles. Note that in both cases $\omega_p = \sqrt{p^2 + m^2} > 0$.

All we used was the fact that the field was complex. Clearly $a_p^\dagger \neq b_p^\dagger$ as these operators create particles of opposite charge. So a global symmetry under phase rotations implies charge which implies complex fields which implies antiparticles. That is,

- Matter coupled to massless spin 1 particles automatically implies the existence of antiparticles, which are particles of identical mass and opposite charge.

This profound conclusion is an inevitable consequence of relativity and quantum mechanics.

To recap, we saw that to have a consistent theory with a massless spin 1 particle we needed gauge invariance. This required a conserved current, which in turn required that charge be conserved. To couple the photon to matter, we needed more than one degree of freedom so we were led to ϕ and $\phi^\star$. Upon quantization, complex scalar fields imply antiparticles. Thus there are many profound consequences of consistent theories of massless spin 1 particles.

2.1 Historical note

Historically, it was the Dirac equation which led to antiparticles. In fact, in 1931 Dirac predicted there should be a particle exactly like the electron except with opposite charge. In 1932 the positron was discovered by Anderson, beautifully confirming Dirac's prediction and inspiring generations of physicists.

Actually, Dirac had an interpretation of antiparticles which sounds funny in retrospect, but was much more logical to him for historical reasons. Suppose we had written

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p^\dagger e^{ipx} + c_p^\dagger e^{-ipx} \right) \tag{9}$$

where both $a_p^\dagger$ and $c_p^\dagger$ are creation operators. Then $c_p^\dagger$ seems to be creating states of negative frequency, or equivalently negative energy. This made sense to Dirac at the time, since there are classical solutions to the Klein-Gordon equation, $E^2 - p^2 = m^2$ with negative energy, so something should create these solutions. Dirac interpreted these negative energy creation operators as removing something of positive energy, and creating an energy hole. But an energy hole in what? His answer was that the universe is a sea full of positive energy states. Then $c_p^\dagger$ creates a hole in this sea, which moves around like an independent excitation.

Then why does the sea stay full, and not collapse to the lower energy configuration? Dirac's explanation for this was to invoke the Fermi exclusion principle. The sea is like the orbitals of an atom. When an atom loses an electron it gets ionized, but it looks like it gained a positive charge. So positive charges can be interpreted as the absence of negative charges, as long as all the orbitals are filled. Dirac argued that the universe might be almost full of particles, so that the negative energy states are the absences of those particles.

It's not hard to see that this is total nonsense. For example, it should work only for fermions, not our scalar field which is a boson. As we have seen, it's much easier to write the creation operator as an annihilation operator to begin with, $c_p^\dagger = b_p$, which cleans everything up immediately. Then the negative energy solutions correspond to the absence of antiparticles, which doesn't require a sea. These days physicists view Dirac's negative frequency excursion and the Dirac sea as historical curiosity, like the aether or phlogiston.

3 Feynman rules for scalar QED

Expanding out the scalar QED Lagrangian we find

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - \phi^\star(\Box + m^2)\phi - ieA_\mu[\phi^\star(\partial_\mu \phi) - (\partial_\mu \phi^\star)\phi] + e^2 A_\mu^2 |\phi|^2 \tag{10}$$

We can read off the Feynman rules from the Lagrangian. The complex scalar propagator is

$$= \frac{i}{p^2 - m^2 + i\varepsilon} \tag{11}$$

This propagator is the Fourier transform of $\langle 0|\phi^\star(x)\phi(0)|0\rangle$ in the free theory. It propagates both ϕ and $\phi^\star$, that is both particles and antiparticles at the same time – they cannot be disentangled.

The photon propagator was calculated in the previous lecture

$$= \frac{-i}{p^2 + i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right] \tag{12}$$

where ξ parametrizes a set of covariant gauges.

Some of the interactions which connect A_μ to ϕ and $\phi^\star$ have derivatives in them, which will give momentum factors in the Feynman rules. To see which momentum factors we get, look back at the quantized fields.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-ipx} + b_p^\dagger e^{ipx}\right) \tag{13}$$

$$\phi^\star(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p^\dagger e^{ipx} + b_p e^{-ipx}\right) \tag{14}$$

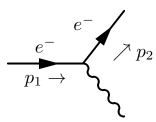
A ϕ in the interaction implies the creation of an antiparticle or the annihilation of a particle at position x. A $\phi^\star$ implies the creation of a particle or the annihilation of an antiparticle. When a derivative acts on these fields, we will pull down a factor of $\pm i p^\mu$ which enters the vertex Feynman rule.

Since the interaction has the form

$$-ieA_\mu\left[\phi^\star(\partial_\mu\phi) - (\partial_\mu\phi^\star)\phi\right] \tag{15}$$

it always has one ϕ and one $\phi^\star$. Each p^μ comes with an i, and there is another i from the expansion of $\exp(i\mathcal{L}_{\text{int}})$, so we always get an overall $(-ie)i^2 = ie$ multiplying whichever $\pm p^\mu$ comes from the derivative. There are 4 possibilities, each one getting a contribution from $A_\mu\phi^\star(\partial_\mu\phi)$ and $-A_\mu\phi(\partial_\mu\phi^\star)$. Calling what a_p annihilates an e^- and what b_p annihilates an e^+ the possibilities are

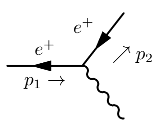
- Annihilate e^- and create e^- – particle scattering



$$= \quad ie(-p_\mu^1 - p_\mu^2)$$

Here, the term $\phi^\star(\partial_\mu\phi)$ gives a $-p_\mu^1$ because the e^- is annihilated by ϕ and the $-\phi(\partial_\mu\phi^\star)$ gives a $-(+p_\mu^2)$ because an e^- is being created by $\phi^\star$. We'll come back to the arrows in a moment.

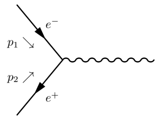
- Annihilate e^+ and create e^+ – antiparticle scattering



$$= \quad ie(p_\mu^1 + p_\mu^2)$$

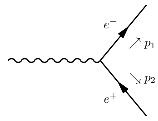
Here, $\phi^\star(\partial_\mu\phi)$ creates the e^+ giving p_μ^2 and $-(\partial_\mu\phi^\star)\phi$ annihilates an e^+ giving $-(-p_\mu^1)$. The next two you can do yourself:

- Annihilate e^- and annihilate e^+ – pair annihilation



$$= \quad i e(-p_\mu^1 + p_\mu^2)$$

- Create a e^- and create e^+ – pair creation



$$= \quad i e(-p_\mu^1 + p_\mu^2)$$

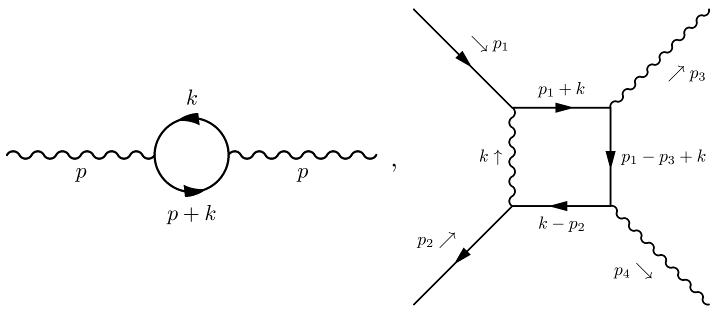
First of all, we see that there are only 4 types of vertices. It is impossible for a vertex to create two particles of the same charge. That is, the Feynman rules guarantee that charge is conserved.

Now let us explain the arrows. In the above vertices, the arrows outside the scalar lines are *momentum-flow arrows*, indicating the direction that momentum is flowing. We conventionally draw momentum flowing from left to right. The arrows superimposed on the lines in the diagram are *particle-flow arrows*. These arrows point in the direction of momentum for particles (e^-) but opposite to the direction of momentum for antiparticles (e^+). If you look at all the vertices, you will see that if the particle-flow arrow points to the right, the vertex gives $-i e p_\mu$, if the particle-flow arrow points to the left, the vertex gives a $+i e p_\mu$. So the particle-flow arrows make the scalar QED Feynman rule easy to remember:

- A scalar QED vertex gives $-i e$ times the sum of the momentum of the particles whose particle-flow arrows point to the right minus the momentum of the particles whose arrows point to the left.

Particle-flow arrows should always make a connected path through the Feynman diagram. For internal lines and loops, whether your lines point left or right is arbitrary; as long as the direction of the arrows is consistent with particle-flow the answer will be the same. If your diagram represents a physical process, external line particle-flow arrows should always point right for particles and to the left for antiparticles.

For loops it is impossible to always have the momentum going to the right. It is conventional in loops to have the momentum flow in the same direction as the charge flow arrows. For uncharged particles, like photons or real scalars, you can pick any directions for the loop momenta you want, as long as momentum is conserved at each vertex. Some examples are:



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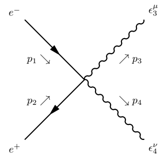
For antiparticles, momentum is flowing backwards to the direction of the arrow. Thus, if particles go forward in time, antiparticles must be going backward in time. This idea was proposed by Stueckelberg in 1941 and independently by Feynman at the famous Poconos conference in 1948 as an interpretation of his Feynman diagrams. The Feynman-Stuckelberg interpretation gives a funny picture of the universe with electrons flying around, bouncing off photons and going back in time, etc. You can have fun thinking about this, but the picture doesn't seem to have much practical application.

Finally, we can't forget that there is another 4-point vertex in scalar QED

$$\mathcal{L}_{\text{int}} = e^2 A_\mu^2 |\phi|^2 \tag{17}$$

This vertex comes from $|D_\mu \phi|^2$, so it is forced by gauge invariance. Its Feynman rule is



$$= 2ie^2 g_{\mu\nu} \tag{18}$$

The 2 comes from the symmetry factor for the 2 A fields. There would not have been a 2 if we had written $\frac{1}{2}e^2 A_\mu^2 |\phi|^2$, but this is not what the Lagrangian gives us. The i comes from the expansion of $\exp(i\mathcal{L}_{\text{int}})$ which we always have for Feynman rules.

3.1 External states

Now we know the vertex factors and propagators for the photon and the complex scalar field. The only thing left in the Feynman rules is how to handle external states. For a scalar field, this is easy – we just get a factor 1. That's because a complex scalar field is just two real scalar fields, so we just take the real scalar field result. The only thing left is external photons.

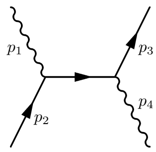
For external photons, recall that the photon field is

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{i=1}^{2} \left(\epsilon_\mu^i(k) a_{k,i} e^{-ikx} + \epsilon_\mu^{i\star}(k) a_{k,i}^\dagger e^{ikx} \right) \tag{19}$$

As far as free states are concerned, which is all we need for S-matrix elements, the photon is just a bunch of scalar fields integrated against some polarization vectors $\epsilon_\mu^i(k)$. Recall that external states with photons have momenta and polarizations, $|k, \epsilon\rangle$, so that $\langle 0|A_\mu(x)|k, \epsilon_i\rangle = \epsilon_\mu^i(k) e^{-ikx}$. This leads to LSZ being modified only by adding a factor of the photon polarization for each external state: ϵ_μ if it's incoming and $\epsilon_\mu^\star$ if it's outgoing.

For example, consider the following diagram:

$i\mathcal{M} =$



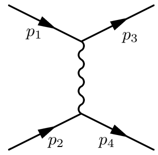
$$= (-ie)\epsilon_\mu^1 (p_2^\mu + k^\mu) \frac{i}{k^2 - m^2 + i\varepsilon} (-ie)(p_3^\nu + k^\nu)\epsilon_\nu^{\star 4},$$

where $k^\mu = p_1^\mu + p_2^\mu$. The first polarization ϵ_μ^1 is the polarization of the photon labeled with p_μ^1. It gets contracted with the momenta $p_2^\mu + k^\mu$ which comes from the $-ieA_\mu[\phi^\star(\partial_\mu\phi) - (\partial_\mu\phi^\star)\phi]$ vertex. The other polarization, ϵ_μ^4, is the polarization of the photon labeled with p_μ^4 and contracts with the second vertex.

4 Scattering in scalar QED

As a first application, let's calculate the cross section for Møller scattering, $e^-e^- \to e^-e^-$ in scalar QED. There are two diagrams. The t-channel diagram (recall the Mandelstam variables s, t and u from Lecture I-7) gives

$i\mathcal{M}_t =$



$$= (-ie)(p_1^\mu + p_3^\mu) \frac{-i[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}]}{k^2} (-ie)(p_2^\nu + p_4^\nu)$$

with

$$k^\mu = p_3^\mu - p_1^\mu \tag{20}$$

But note that

$$k^\mu(p_1^\mu + p_3^\mu) = (p_3^\mu - p_1^\mu)(p_3^\mu + p_1^\mu) = p_3^2 - p_1^2 = m^2 - m^2 = 0 \tag{21}$$

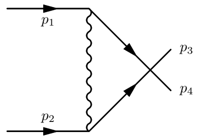
So this simplifies to

$$\mathcal{M}_t = e^2 \frac{(p_1^\mu + p_3^\mu)(p_2^\mu + p_4^\mu)}{t}$$

and the ξ-dependence has vanished. We expected this to happen, by gauge invariance, and now we have seen that it does indeed happen. Note that $k_\mu(p_1^\mu + p_3^\mu) = 0$ is just $k_\mu J_\mu = 0$, so this is a direct consequence of current conservation.

The u-channel gives

$$i\mathcal{M}_u =$$



$$= (-ie)(p_1^\mu + p_4^\mu) \frac{-i[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}]}{k^2} (-ie)(p_2^\nu + p_3^\nu),$$

where $k^\mu = p_4^\mu - p_1^\mu$. In this case,

$$k^\mu(p_1^\mu + p_4^\mu) = p_4^2 - p_1^2 = 0 \tag{22}$$

so that

$$\mathcal{M}_u = e^2 \frac{(p_1^\mu + p_4^\mu)(p_2^\mu + p_3^\mu)}{u}$$

Thus the cross section for scalar Møller scattering is

$$\frac{d\sigma(e^- e^- \to e^- e^-)}{d\Omega} = \frac{e^4}{64\pi^2 E_{\text{cm}}^2} \left[\frac{(p_1^\mu + p_3^\mu)(p_2^\mu + p_4^\mu)}{t} + \frac{(p_1^\mu + p_4^\mu)(p_2^\mu + p_3^\mu)}{u} \right]^2 \tag{23}$$

$$= \frac{\alpha^2}{4s} \left[\frac{s-u}{t} + \frac{s-t}{u} \right]^2 \tag{24}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant.

5 The Ward identity and gauge invariance

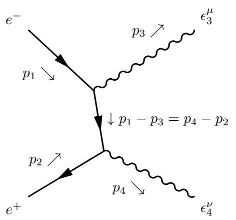
We saw in the previous example that the matrix elements for a particular amplitude in scalar QED were independent of the gauge parameter ξ. The photon propagator is

$$i\Pi_{\mu\nu} = \frac{-i\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2} \right]}{p^2 + i\varepsilon} \tag{25}$$

A general matrix element involving an internal photon will be $M_{\mu\nu}\Pi_{\mu\nu}$ for some $M_{\mu\nu}$. So gauge-invariance, which in this context means ξ-independence, requires $M_{\mu\nu} p_\mu p_\nu = 0$. Gauge invariance in this sense is closely related to the Ward identity which required $p_\mu M_\mu = 0$ if the matrix element involving an on-shell photon is $\epsilon_\mu M_\mu$. Both gauge invariance and the Ward identity hold for any amplitude in scalar QED. However it is somewhat tedious to prove this in perturbation theory. In this section, we will give a couple of examples illustrating what goes into the proof, with the complete non-perturbative proof postponed until Lecture II-7 after path integrals are introduced.

As an non-trivial example where the Ward identity can be checked, consider the process $e^+ e^- \to \gamma\gamma$. A diagram contributing to this is

$$i\,\mathcal{M}_t =$$



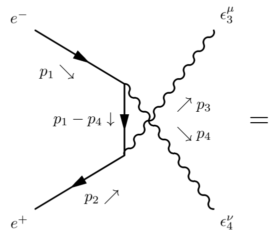
$$= (-ie)^2 \frac{i(2p_1^\mu - p_3^\mu)(p_4^\nu - 2p_2^\nu)}{(p_1 - p_3)^2 - m^2} \epsilon_3^{\star\mu} \epsilon_4^{\star\nu} \tag{26}$$

Using only that the electron is on shell (not assuming $p_3^2 = p_4^2 = p_3 \cdot \epsilon_3 = p_4 \cdot \epsilon_4$), this simplifies slightly to

$$\mathcal{M}_t = e^2 \frac{(p_3 \cdot \epsilon_3^\star - 2p_1 \cdot \epsilon_3^\star)(p_4 \cdot \epsilon_4^\star - 2p_2 \cdot \epsilon_4^\star)}{p_3^2 - 2p_3 \cdot p_1} \tag{27}$$

The crossed diagram gives the same thing with $1 \leftrightarrow 2$ (or equivalently $3 \leftrightarrow 4$)

$$i\mathcal{M}_u =$$



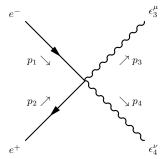
$$= i e^2 \frac{(p_3 \cdot \epsilon_3^\star - 2p_2 \cdot \epsilon_3^\star)(p_4 \cdot \epsilon_4^\star - 2p_1 \cdot \epsilon_4^\star)}{p_3^2 - 2p_3 \cdot p_2} \tag{28}$$

To check whether the Ward identity is satisfied with just these two diagrams, we replace $\epsilon_3^{\star\mu}$ with p_3^μ giving

$$\mathcal{M}_t + \mathcal{M}_u = e^2[p_4 \cdot \epsilon_4^\star - 2p_2 \cdot \epsilon_4^\star + p_4 \cdot \epsilon_4^\star - 2p_1 \cdot \epsilon_4^\star] = 2e^2 \epsilon_4^{\star\mu}(p_4^\mu - p_2^\mu - p_1^\mu) \tag{29}$$

which is in general non-zero. The resolution is the missing diagram involving the 4-point vertex:

$$i\mathcal{M}_4 =$$



$$= 2 i e^2 g_{\mu\nu} \epsilon_3^{\star\mu} \epsilon_4^{\star\nu} \tag{30}$$

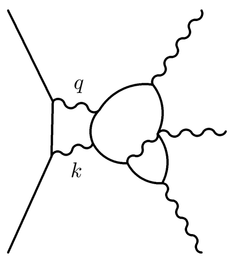
Thus, replacing $\epsilon_3^{\star\mu}$ with p_3^μ and summing all the diagrams, we have

$$\mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_4 = 2e^2 \epsilon_4^{\star\mu}(p_4^\mu - p_2^\mu - p_1^\mu + p_3^\mu) = 0 \tag{31}$$

and the Ward identity is satisfied.

The above derivation did not actually require us to use that the photons are on-shell or massless. That is, we did not apply any of $p_3^2 = p_4^2 = \epsilon_3^\star \cdot p_3 = \epsilon_4^\star \cdot p_4 = 0$. Thus, the Ward identity would be satisfied even if the external photon states were not physical. For example if they were in a loop. In fact, that's exactly what we need for gauge invariance, so the same calculation can be used to prove ξ-independence.

To prove gauge invariance, we need to consider internal photon propagators, for example in a diagram like



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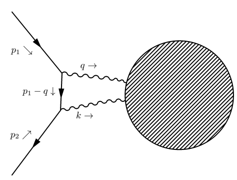
Let us focus on showing ξ independence for the propagators labeled q and k. For this purpose, the entire right side of the diagram (or the left side) can be replaced by a generic tensor $X_{\alpha\beta}$ depending only on the virtual momenta of the photons entering it. The index α will contract with the q photon propagator, $\Pi_{\mu\alpha}(q)$ and β with the k photon propagator, $\Pi_{\nu\beta}(k)$. Diagrammatically, this means

$$i\mathcal{M}_t =$$



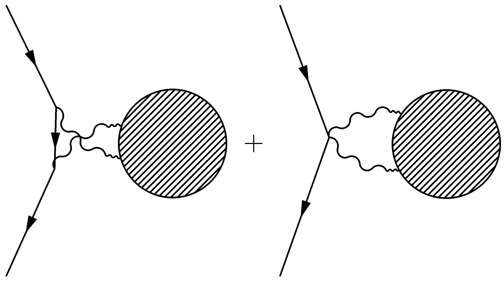
(33)

which is very closely related to the t-channel diagram above, Eq. (26). The integral can be written in the form

$$\mathcal{M}_t = \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \delta^4(p_1 + p_2 - k - q) e^2 \frac{(q^\mu - 2p_1^\mu)(k^\nu - 2p_2^\nu)}{q^2 - 2q \cdot p_1} \Pi_{\mu\alpha}(q) \Pi_{\nu\beta}(k) X_{\alpha\beta}(q, k) \tag{34}$$

where we have inserted an extra integral over momentum and an extra δ-function to keep the amplitude symmetric in q and k. Comparing with Eq. (26), the polarization vectors ϵ_3^μ and ϵ_4^ν have been replaced by contractions with the photon propagators and $p_3 \to q$ and $p_4 \to k$. Replacing $\Pi_{\mu\alpha}(q)$ by $\xi q_\mu q_\alpha$ we see that the result does not vanish, implying that this diagram alone is not gauge invariant.

To see gauge invariance, we need to include all the diagrams which contribute at the same order. This includes the u-channel diagrams and the one involving the 4-point vertex



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Adding these graphs, we get same sum as before

$$\mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_4 = e^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \delta^4(p_1 + p_2 - k - q) \tag{36}$$

$$\times \left[\frac{(q^\mu - 2p_1^\mu)(k^\nu - 2p_2^\nu)}{q^2 - 2q \cdot p_1} + \frac{(q^\mu - 2p_2^\mu)(k^\nu - 2p_1^\nu)}{q^2 - 2q \cdot p_2} + 2g^{\mu\nu} \right] \Pi_{\mu\alpha}(q) \Pi_{\nu\beta}(k) X_{\alpha\beta}(q, k) \tag{37}$$

Now if we replace $\Pi_{\mu\alpha}(q) \to \xi q_\mu q_\alpha$ we find

$$\mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_4 \to 2\xi e^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \delta^4(p_1 + p_2 - k - q)(k^\nu - p_2^\nu - p_1^\nu + q^\nu) q^\alpha \Pi_{\nu\beta}(k) X_{\alpha\beta}(q, k)$$

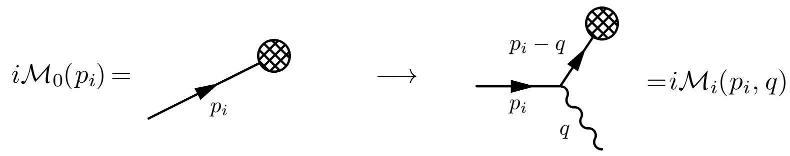
which exactly vanishes. Thus gauge invariance holds in this case. The case of a photon attaching to a closed scalar loop is similar and you can explore it in Problem ??.

A general diagrammatic proof involves arguments like this, generalized to an arbitrary number of photons and possible loops. The only challenging part is keeping track of the combinatorics associated with the different diagrams. Some examples can be found in [Zee] and in [Peskin and Schroeder]. The complete diagrammatic proof is actually easier in real QED (with fermions) than in scalar QED, since there is no 4-point vertex in QED. As mentioned above, we will give a complete non-perturbative proof of both gauge invariance and the Ward identity in Lecture II-8.

6 Lorentz invariance and charge conservation

There is a beautiful and direct connection between Lorentz invariance and charge conservation that bypasses gauge invariance completely. What we will now show is that a theory with a massless spin-1 particle automatically has an associated conserved charge. This profound result, due to Weinberg [], does not require a Lagrangian description: it only uses little-group invariance and the fact that for a massless field one can take the soft limit.

Imagine we have some diagram with lots of external legs and loops and things. Say the matrix element for this process is $\mathcal{M}_0$. Now tack on an outgoing photon of momentum q_μ and polarization ϵ_μ onto an external leg. For simplicity, we take ϵ_μ real to avoid writing $\epsilon_\mu^\star$ everywhere. Let's first tack the photon onto leg i, which we take to be an incoming e^-.



This modifies the amplitude to

$$\mathcal{M}_i(p_i, q) = (-ie)\frac{i[p_i^\mu + (p_i^\mu - q^\mu)]}{(p_i - q)^2 - m^2}\epsilon^\mu \mathcal{M}_0(p_i - q) \tag{38}$$

We can simplify this using $p_i^2 = m^2$ and $q^2 = 0$ in the denominator, since the electron and photon are on-shell, and $q_\mu \epsilon_\mu = 0$ in the numerator since the polarizations of physical photons are transverse to their own momenta. Then we get

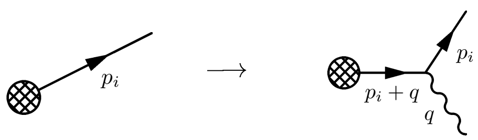
$$\mathcal{M}_i(p_i, q) = -e\frac{p_i \cdot \epsilon}{p_i \cdot q}\mathcal{M}_0(p_i - q) \tag{39}$$

Now take the soft limit. By soft we mean that $|q \cdot p_i| \ll |p_j \cdot p_k|$ for all the external momenta p_i, not just the one we modified. Then $\mathcal{M}_0(p_i - q) \approx \mathcal{M}_0(p_i)$, where $\approx$ indicates the soft limit. Note that that photons attached to loop momenta in the blob in $\mathcal{M}_0$ are subdominant to photons attached to external legs, since the loop momenta are off-shell and hence the associated propagators are not singular as $q \to 0$. That is, photons coming off loops cannot give $\frac{1}{p_i \cdot q}$ factors. Thus in the soft limit, the dominant effect comes only from diagrams where photons attached to external legs. We must sum over all such diagrams.

If the leg is an incoming e^+, we would get

$$\mathcal{M}_i(p_i, q) \approx e\frac{p_i \cdot \epsilon}{p_i \cdot q}\mathcal{M}_0(p_i) \tag{40}$$

where the sign flip comes from the charge of the e^+. If the leg is an outgoing electron, it's a little different. The photon is still outgoing, so we have



and the amplitude is modified to

$$\mathcal{M}_i(p_i, q) = (-ie)\frac{i[p_i^\mu + (p_i^\mu + q^\mu)]}{(p_i + q)^2 - m^2}\epsilon_\mu \mathcal{M}_0(p_i + q) \approx e\,\frac{p_i \cdot \epsilon}{p_i \cdot q}\mathcal{M}_0(p_i) \tag{41}$$

Similarly for an outgoing positron, we would get another sign flip and

$$\mathcal{M}_i(p_i, q) \approx -e\frac{p_i \cdot \epsilon}{p_i \cdot q}\mathcal{M}_0(p_i) \tag{42}$$

If we had many different particles with different charges, these formulas would be the same but the charge Q_i would appear instead of ± 1.

Summing over all the particles we get

$$\mathcal{M} \approx e\mathcal{M}_0\left[\sum_{\text{incoming}} Q_i \frac{p_i \cdot \epsilon}{p_i \cdot q} - \sum_{\text{outgoing}} Q_i \frac{p_i \cdot \epsilon}{p_i \cdot q}\right] \tag{43}$$

where Q_i is the charge of particle i.

Here comes the punchline. Under a Lorentz transformation, $\mathcal{M}(p_i, \epsilon) \to \mathcal{M}(p_i', \epsilon')$ where p_i' and ϵ' are the momenta and polarization in the new frame. Since $\mathcal{M}$ must be Lorentz invariant, the transformed $\mathcal{M}$ bust be the same. However, polarization vectors do not transform exactly like 4-vectors. As we showed explicitly in the last lecture, there are certain Lorentz transformations for which q_μ is invariant and

$$\epsilon_\mu \to \epsilon_\mu + q_\mu \tag{44}$$

These transformations are members of the little group so the basis of polarization vectors does not change. Since there is no polarization proportional to q_μ, there does not exist a physical polarization ϵ_μ' in the new frame which is equal to the transformed ϵ_μ. Therefore $\mathcal{M}$ has to change, violating Lorentz invariance. The only way out is if the q_μ term does not contribute. In terms of $\mathcal{M}$, the little group transformation effects

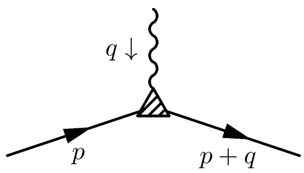
$$\mathcal{M} \to \mathcal{M} + e\mathcal{M}_0\left[\sum_{\text{incoming}} Q_i - \sum_{\text{outgoing}} Q_i\right] \tag{45}$$

and therefore the only way for $\mathcal{M}$ to be Lorentz invariant is

$$\sum_{\text{incoming}} Q_i = \sum_{\text{outgoing}} Q_i \tag{46}$$

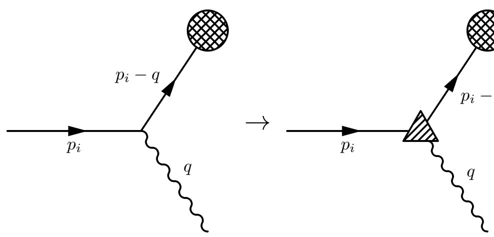
which says that charge is conserved. This is a sum over all of the particles in the original $\mathcal{M}_0$ diagram, without the soft photon. Since this process was arbitrary, we conclude that charge must always be conserved.

Although we used the form of the interaction in scalar QED to derive the above result, it turns out this result is completely general. For example, suppose the photon had an arbitrary interaction with ϕ. Then the Feynman rule for the vertex could have arbitrary dependence on momenta:



$$= -ie\Gamma_\mu(p, q) \tag{47}$$

The vertex must have a μ index to contract with the polarization, by Lorentz invariance. Furthermore, also by Lorentz invariance since the only 4-vectors available are p^μ and q^μ we must be able to write $\Gamma^\mu = 2p^\mu F(p^2, q^2, p \cdot q) + q^\mu G(p^2, q^2, p \cdot q)$. Functions like F and G are sometimes called **form factors**. In scalar QED, $F = G = 1$. Since $q^\mu \epsilon_\mu = 0$ we can discard G. Moreover, since $p^2 = m^2$ and $q^2 = 0$, the form factor can only be a function of $\frac{p \cdot q}{m^2}$ by dimensional analysis, so we write $\Gamma^\mu = 2p^\mu F\left(\frac{p \cdot q}{m^2}\right)$. Now we put this general form into the above argument, so that



$$\approx -eF_i(0)\frac{p_i \cdot \epsilon}{p_i \cdot q}\mathcal{M}_0(p_i) \tag{48}$$

$F_i(0)$ is the only relevant value of $F_i(x)$ in the soft limit. We have added a subscript i on F since F_i can be different for different particles i. Although $F_i(x)$ does not have to be an analytic function, its limit as $x \to 0$ should be finite or else the matrix element for emitting a soft photon would diverge. Then Eq. (45) becomes

$$\mathcal{M} \to \mathcal{M} - e\mathcal{M}_0\left[\sum_{\text{incoming}} F_i(0) - \sum_{\text{outgoing}} F_i(0)\right] \tag{49}$$

Thus we get the same result as before, and moreover produce a general definition of the charge $Q_i = -F_i(0)$. (This definition will re-emerge in the context of renormalization, in Lecture III-4.)

Thus the connection between a massless spin-1 particle and conservation of charge is completely general. In fact, the same result holds for charged particles of any spin. In fact, the $\frac{p\cdot\epsilon}{p\cdot q}$ form of the interaction between light and matter in the soft limit is universal and spin-independent (it is called an **Eikonal interaction**, and will be discussed again in Lecture IV-12). The conclusion is very general:

- **Massless spin-1 particles imply conservation of charge**

Note that masslessness of the photon was important in two places: that there are only two physical polarizations, and that we can take the soft limit with the photon on-shell.

"What's the big deal?" you say, "we knew that already." But in the derivation from the previous lecture, we had to use gauge invariance, gauge fix, isolate the conserved current, etc. Those steps were all artifacts of trying to write down a nice simple Lagrangian. The result we just derived doesn't require Lagrangians or gauge invariance at all. It just uses that a massless particle of spin-1 has two polarizations and the soft limit. Little group scaling was important, but only to the extent that the final answer had to be a Lorentz-invariant function of the polarizations and momenta 4-vectors. The final conclusion, that charge is conserved, doesn't care that we embedded the two polarizations in a 4-vector ϵ_μ. It would be true even if we only used on-shell helicity amplitudes (an alternative proof without polarization vectors is given in Lecture IV-3).

To repeat, this is a non-perturbative statement about the physical universe, not a statement about our way of doing computations, like gauge invariance and the Ward identities are. Proofs like this are rare and very powerful. In Problem 3 you can show in a similar way that, when multiple massless spin-1 particles are involved, the soft limit forces them to transform in the adjoint representation of a Lie group. We now turn to the implications of the soft limit for massless particles of integer spin greater than 1.

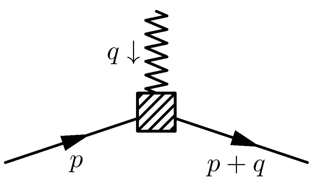
6.1 Lorentz invariance for spin 2 and higher

A massless spin-2 field has 2 polarizations $\epsilon_{\mu\nu}^i$ which rotate into each other under Lorentz transformations, but also into $q_\mu q_\nu$. There are little group transformations which send

$$\epsilon_{\mu\nu} \rightarrow \epsilon_{\mu\nu} + \Lambda_\mu q_\nu + \Lambda_\nu q_\mu + \Lambda q_\mu q_\nu \tag{50}$$

where these Λ_μ vectors have to do with the explicit way the Lorentz group acts, which we don't care about so much. Thus, any theory involving a massless spin-2 field should satisfy a Ward identity: if we replace even one index of the polarization tensor by q_μ the matrix elements must vanish. The spin-2 polarizations can be projected out of $\epsilon_{\mu\nu}$ as the transverse-traceless modes: $q_\mu \epsilon_{\mu\nu} = \epsilon_{\mu\mu} = 0$.

What do the interactions look like? As in the scalar case, they don't actually matter, and we can write a general interaction as



$$= -i\Gamma^{\mu\nu}(p,q) = -2ip^\mu p^\nu \tilde{F}\left(\frac{p\cdot q}{m^2}\right) \tag{51}$$

where $\tilde{F}(x)$ is some function, different in general from the spin 1 form factor $F(x)$. The μ and ν indices on $\Gamma^{\mu\nu}$ will contract with the indices of the spin 2 polarization vector $\epsilon_{\mu\nu}$.

Taking the soft limit and adding up diagrams for incoming and outgoing spin 2 particles, we find

$$\mathcal{M} = \mathcal{M}_0 \left[\sum_{\text{incoming}} \tilde{F}_i(0) \frac{p_i^\mu}{p_i \cdot q} \epsilon_{\mu\nu} p_\nu^i - \sum_{\text{outgoing}} \tilde{F}_i(0) \frac{p_i^\mu}{p_i \cdot q} \epsilon_{\mu\nu} p_i^\nu \right] \tag{52}$$

which is similar to what we had for spin 1, but with an extra factor of p_ν^i in each sum.

By Lorentz invariance, little group transformations like those in Eq. (50) imply that this should vanish if $\epsilon_{\mu\nu} = q_\mu \Lambda_\nu$ for any Λ_ν. So, writing $\kappa_i \equiv \tilde{F}_i(0)$, which is just a number for each particle, we find

$$\mathcal{M}_0 \Lambda_\nu \left[\sum_{\text{incoming}} \kappa_i p_i^\nu - \sum_{\text{outgoing}} \kappa_i p_i^\nu \right] = 0 \tag{53}$$

which implies

$$\sum_{\text{incoming}} \kappa_i p_i^\nu = \sum_{\text{outgoing}} \kappa_i p_i^\nu \tag{54}$$

In other words, the sum of $\kappa_i p_i^\nu$ is conserved. But we already know, by momentum conservation, that the sum of p_i^μ is conserved. So, for example, we can solve for p_1^μ in terms of the others. If we add another constraint on the p_i^μ then there would be a different solution for p_1^μ, which is impossible unless all the p_i^μ are zero. The only way we can have nontrivial scattering is for all the charges to be the same

$$\kappa_i = \kappa \quad \text{for all } i \tag{55}$$

But that's exactly what gravity does! All particles gravitate with the same strength $\kappa_i \equiv \frac{1}{M_P} = \sqrt{G_n}$. In other words gravity is universal. So,

- **Massless spin-2 particles imply gravity is universal**.

We can keep going. For massless spin 3 we would need

$$\sum_{\text{incoming}} \beta_i p_\nu^i p_\mu^i = \sum_{\text{outgoing}} \beta_i p_\nu^i p_\mu^i \tag{56}$$

where $\beta_i = \hat{F}_i(0)$ for some generic spin 3 form factor $\hat{F}_i\left(\frac{p \cdot q}{m^2}\right)$. For example, the $\mu = \nu = 0$ component of this says

$$\sum_{\text{incoming}} \beta_i E_i^2 = \sum_{\text{outgoing}} \beta_i E_i^2 \tag{57}$$

that is the sum of the squares of the energies times some charges are conserved. That's way too constraining. The only way out is if all the charges are 0, which is a boring, non-interacting theory of free massless spin 3 field. So,

- **There are no interacting theories of massless particles of spin greater than 2.**

And in fact, no massless particles with spin $>$ 2 have ever been seen. (Massive particles of spin >2 are plentiful.)