Central Limit Theorem and Beyond

- CLT ↔ conventional wisdom on random processes, in depth
- Beyond: - Gaussian from CLT, as special case of Levy Stable Distribution
  - Levy Distribution, an Introduction
  - Levy Process - generalizing random walk

Central Limit Theorem: (DeMoivre, Laplace, Gauss)
- Consider a sum of \( \mathcal{N} \)-independent random variables, increments...

\[ AX_1, AX_2, \ldots, AX_n \quad (n \gg 1) \]

Let sum \( X_n = \sum_i AX_i \)
steps i.i.d.
- Identically, independently distributed
  - Each \( \Delta X_i \): \( \langle \Delta X_i \rangle = 0 \)
    \( \langle \Delta X_i^2 \rangle = \sigma_i^2 \)
  - Variance of step distribution converges
  - Only variance required
  - then \( \sigma^2 = \sum_i \sigma_i^2 \)

CLT ⇒
\[
\text{PDF} \left( x_n \right) = \frac{1}{\left(2\pi\sigma^2\right)^{\frac{1}{2}}} e^{-\frac{x_n^2}{2\sigma^2}}
\]
\( n \gg 1 \)
- PDF sum → Gaussian

- Key Points / Buried Bodies
  1. \( \Delta X_i \) (likewise), sum not dominated by few (e.g., intermittent)
  2. Finite variance of step increment \( \langle \Delta X_i^2 \rangle < \infty \)
b1) What of higher moments? 
\[ \langle \Delta x_i^2 \rangle < \infty \neq \langle \Delta x_i^4 \rangle < \infty \]

\[ \Rightarrow \text{large fluctuations can induce heavy tails} \]

\[ \Rightarrow \text{fat tail} \]

b2) Quantity?

Observations:

- CLT states (effectively) that (given conditions satisfied),

\[ X_i \rightarrow \text{statistically independent and consistent with CLT} \]

Then if \[ X_i, Y_i \] are series to be summed, which follow CLT conditions.
then \((ax_i + b) + (a'y_i + b')\) 
\[= a''z_i + b''\]

is also Gaussian distributed, i.e. follows CLT ("L- stobility")

here: \(a, b, j, a', b\) all > 0 and not stochastic

In simple terms:

\(\Rightarrow\) Adding two Gaussian distributed series yields a sum which is Gaussian distributed.

\(\Rightarrow\) CLT "Gaussian modulo conditions, is an attractor in function space"

More generally: A class of distributions exist which are
L-stable (L for Paul Levy)

c.a have property that if t tw
series distributed, sum
is also distributed similarly.

The Message:

The reduced Gaussian of CLT
is merely one particular case
of an L-stable distribution, and
the only one with finite variance.

Many elements in class of L-stable
attractors on function space,

Family of allowed distributions
is larger than you thought...

To understand: First re-visit CLT!
Proving the C.L.T.

General ideas:

- Markov process → Chapman - Kolmogorov E2n

- Convolution

- Convolution → Product of F.T.

  → "Generating" or "characteristic" Function

Point: Fourier Transform of step probability is more significant than probability (and useful)

So C - K E2n:

\[ P_N(x) = \sum \text{dy} \, P_N(y) \, P_N(x \mid x, y) \]

don't expand ----
then,

if F.T. and noting F.T. (convolution) = \( \mathcal{F} \) F.T., i.e., Fourier transform of convolution = product of functions convolved.

then, \( N \)-step C-LT:

\[
\tilde{P}_N(k) = \tilde{p}_1(k) \tilde{p}_2(k) \cdots \tilde{p}_N(k) = \sum_{n=1}^{N} \tilde{p}_n(k)
\]

\[
\mathcal{P}_N(x) = \mathcal{F} \left( \mathcal{F}^{-1} \left( \mathcal{F}^{-1} \left( \sum_{n=1}^{N} \tilde{p}_n(k) \right) \right) \right)
\]

applies for identical steps.
2. can also define moments:

\[ m_n = \int dx \ x^n \ p(x) \]

\[ \langle x \rangle = m_1 \]

\[ \langle x^2 \rangle = m_2 \]

\[ \langle x^n \rangle = m_n \]

\[ \hat{p}(k) = \sum_{n=0}^{\infty} (-i)^n \frac{k^n}{n!} m_n \]

\[ = \int e^{-ikx} \ dx \ p(x) \]

\[ = \int dx \ (1 - ikx + \frac{(ikx)^2}{2} + ...) \ p(x) \]

\[ = \int dx \ \sum_{n=0}^{\infty} (-i)^n \frac{k^n}{n!} x^n \ p(x) \]

\[ m_n = \int \frac{\partial^n p(x)}{\partial k^n} \]

useful identity: \[ n^{th} \ \text{moment} \quad \text{is} \quad \text{n}^{th} \ \text{derivative of generating fn} \]
\[ \hat{\psi}(k) = 1 - i \mathcal{M}_1(k) - \frac{1}{2} \mathcal{M}_2(k^2) + \ldots \]

easily generalized to higher dimensions.

\( \text{3 Cumulants} \)

- i.e. nonlinear combinations of moments

\[ \psi(k) = \ln \hat{\rho}(k) \]

\[ \hat{\rho}(k) = \int \frac{e^{i k \cdot x}}{(2\pi)^{d/2}} d^d k \hat{\rho}(k) \]

\[ = \int e^{i k \cdot x + \psi(k)} \frac{d^d k}{(2\pi)^{d/2}} \]

expand:

\[ \psi(k) = -i C_1(k) - \frac{1}{2} C_2(k^2) + \ldots \]

\( C_1 = m_1 \)

\( C_2 = m_2 - m_1^2 = \sigma^2 \)

\{ \text{series of cumulants} \}

\{ \text{cumulants} \}

et cetera.
If exist, one has from moments:

\[ \hat{c}_n \mathbb{E}_n \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \delta_{n,0} \]

Now assuming independent, independently distributed (IID) additive steps, cumulants

\[ A_N(x) = \sum \frac{e^{ikx}}{2\pi} A_N(k) \]

\[ = \sum \frac{e^{ikx}}{2\pi} (\mathbb{E}(x))^N \]

\[ = \sum \frac{e^{ikx}}{2\pi} (e^{\mu x})^N \]

\[ = \sum \frac{e^{ikx}}{2\pi} e^{N\mu x} \]

\[ \mathcal{F}_N(x) = N \mathcal{F}(x) \]
δ_0 \text{ for C.L.T.}

Consider \( N \to \infty \)

(very more to the \( \delta \))

\[
\rho_N(x) = \sum_{n=1}^{\infty} \frac{\hat{\rho}_n(x)}{\sqrt{2\pi} n} \cdot \frac{\hat{\rho}_n(x)}{\sqrt{2\pi} n} \cdot \frac{e^{i k x}}{\sqrt{2\pi}}
\]

\[
= \int_{-\infty}^{\infty} e^{i k x} \frac{e^{i k x}}{\sqrt{2\pi}} \cdot \psi(k)^N
\]

\[
= \int_{-\infty}^{\infty} e^{i k x} \frac{e^{i k x}}{\sqrt{2\pi}} \cdot e^{N \psi(k)}
\]

i.e. additivity: \( \rho_N = N \psi(k) \)

\[
\psi(k) = -i c k - k^2 c_2
\]

\[
\rho_N(x) = \int_{-\infty}^{\infty} e^{i k x} \frac{e^{i k x}}{\sqrt{2\pi}} \cdot e^{N \psi(k)}
\]

For \( N \to \infty \) only the region near \( k = 0 \) contributes (Laplace's Method)

\( \geq \) only low order cumulants contribute / determine \( \rho_N(x) \)

N.B. Fundamental reasoning for truncating monomers - Moyal
\[ \text{NC}_0 = N_{\text{NC}_0} \leq \frac{\lambda}{c} \Rightarrow n = \frac{c}{\lambda} \]

\[ P_n(x) = \frac{e^{-\frac{\lambda}{c}x} \left( \frac{\lambda}{c}x \right)^n}{n!} \]

\[ P_n(x) = \int_{\frac{\lambda}{c}x}^{\infty} e^{-\lambda y} y^n dy \]

\[ \text{NC}_0 \leq \frac{\lambda}{c} \Rightarrow n \leq \frac{c}{\lambda} \]

\[ \int_{\frac{\lambda}{c}x}^{\infty} e^{-\lambda y} y^n dy \]

\[ \text{and even:} \]

\[ \text{if} c \leq \lambda \]

\[ P_n(x) = \int_{\frac{\lambda}{c}x}^{\infty} e^{-(\lambda - \frac{\lambda}{c})y} y^n dy \]

\[ \text{if} c = \lambda \]

\[ \text{if} c > \lambda \]
\[ P(x, t) = \frac{1}{(\pi t)^{1/2}} \exp \left[ -\frac{x^2}{2t} \right] \]

\[ \Rightarrow \text{C.L.T.} \]

A few points:

- No questions asked about higher moments, for \( N \to \infty \).

- These need not be well behaved, and induce \underline{Fat tails}.

\[ P(x) = \frac{1}{1+x^2} \]

has \( \langle x^2 \rangle \to \infty \), \underline{so C.L.T. not apply}.

but \( P(x) = \frac{2}{\sqrt{\pi}} \left(1 + x^4\right)\)

\( \langle x^2 \rangle < \infty \)

meets C.L.T. criteria, \underline{but kurtosis diverges}

\( \Rightarrow \text{Fat Tail} \)
- Can show,
  - Gaussian ended (fat tail + probability conserved = erode central Gaussian)
  - large \( X \) (how large is "large"?)

\[
P_N(CX) \sim N(A/\sqrt{X})^4
\]
(power law, not Gaussian)

N.B. - Refs: Chandrasekher Review
  - Hikos et. al.
  - Hughes, B.D.

  or any book...

- M.I.T. OCW 18.366
  ("Random Walks and Diffusion")

- Physics 235, Spring 19
  (Note write-ups, supplementary)

N.B. Issue of Fat Tail behavior within
CLT is good paper topic.
Levy Distributions

Observe: A property of diffusion $\Rightarrow$ Self-Similarity

$D = \frac{\langle \Delta x^2 \rangle}{\Delta t}$

$\Delta x = x - x'$

$\Delta t = \Delta \xi$

$D' = x^{-2} \langle \Delta x^2 \rangle = 0$

$\frac{\partial^2}{\partial t}$

If: $\beta = \chi^2$

What is the class of self-similar distributions which are L-stable and normalizable?

Now, $x_i \to$ random variable

$x_N = \sum_{i=1}^{N} x_i$

$\hat{\beta}(\chi)$ function:

$\hat{\beta}(\chi) = [P(\chi)]^\chi$
Rescale: \[ Z_n = x_n / a_n \]

\[ \text{pdf}(Z_n) = F_{n_0}(X) / a_n \]

\[ x = x_n / a_n \]  

\[ P(c, z) P(c, z) = P(c, z) \]

\[ \hat{F}_n(c, z) = \hat{p}_n(z) \]

Now seek attractors in function space, so:

\[ F_n(z) \rightarrow \hat{F}(z) \quad \text{as } n \rightarrow \infty \]

Let \( \lim_{m \rightarrow \infty} \frac{a_n}{a_m} = c_n \).

(Uses\ Stability)

Then have condition for function \( \hat{F}(z) \) as limiting case

\[ \hat{F}(z, c_n) = [\hat{F}(z)]^{c_n} \]

\[ \text{self-similarity} \]
So need solve
\[ \hat{\Psi} (k \mu(x)) = (E(k))^\lambda \]

\[ \Psi = \ln E(k) \]
\[ \Psi (k \mu(x)) = \lambda \Psi (k) \]
with \( \mu(x=1) = 1 \).

\[ \frac{\partial}{\partial \gamma} \psi (k \mu(x)) = \Psi (k) \]
\[ \gamma \mu \psi = \psi \]
\[ \frac{1}{\partial \lambda} = \lambda / \mu \]
\[ 1 / \mu / \partial \lambda \]

\( \psi \)

Power law for \( \psi \)
\[ \psi (k) = \begin{cases} v_1 | k |^{\lambda}, & k > 0 \\ v_2 | k |^{\lambda}, & k < 0 \end{cases} \]
Can show in more detail: (Hughes)

\[ \hat{F}(k) = \exp \left[ -a |k|^{\alpha} (1 - i \beta \tan(\theta/2)) \right] \]

\[ \text{Strewness} \]

take $\beta = 0$ \rightarrow \text{Levy Distribution}

\[ L_\alpha(a, k) = \hat{F}(k) = \exp(-a |k|^{\alpha}) \]

$\alpha = 2 \Rightarrow \hat{F}(k) = \exp(-a k^2) \rightarrow L_2$

Gaussian.

$\alpha = 2$ it self-similar (attractor)

$L$-stable with \text{normalizable} \ 2nd \ \text{moment} \ \text{CC-L.T. Case} \ (\text{only if})$

Can show $\alpha = 2$ is max. $\alpha$.

$\alpha = 1 \Rightarrow \hat{F}(k) = C^{-a |k|} \Rightarrow \text{Cauchy, Lorentzian}$

$P(x) = \frac{1}{\pi(\alpha^2 + x^2)}$