1 Concepts

1. Elasticity:

Atoms in a solid are arranged in a crystalline lattice structure, where each atom is within an equilibrium distance with its neighbor. We model the atomic bonds of solids as extremely stiff springs, which makes the material seemingly rigid when small forces are applied. When a force is applied to solid rod, \( F_{\text{app}} \), the atoms are displaced from equilibrium and the rod stretches out by \( \Delta L \).

The rod exerts a corresponding restoring force as described by Hooke’s Law. Therefore, the applied force \( F_{\text{app}} \) necessary to stretch the rod by \( \Delta L \) is

\[
F_{\text{app}} = k_{\text{eff}} \Delta L
\]

where \( k_{\text{eff}} \) is the effective spring constant of the entire rod. Note that this linear relationships holds up until a certain applied force strength. We call this the elastic limit. However, the effective spring constant of the rod \( k_{\text{eff}} \) depends on both its composition (i.e. iron, steel, concrete) and its geometry (i.e. length and cross-sectional area). The geometric properties of the rod determine how many molecular bonds are in the solid.

The elasticity of a material directly relates to the spring constant of each molecular bond, which only depends on the rod’s composition. The force pulling each bond is proportional to \( \frac{F_{\text{app}}}{A} \), where \( A \) is the cross-sectional area of the rod. The amount that each bond stretches is proportional to \( \frac{\Delta L}{L} \), where \( L \) is the length of the rod. Putting this together we have,

\[
\frac{F_{\text{app}}}{A} = Y \frac{\Delta L}{L}
\]

where \( Y \) is the Young’s Modulus, which is the constant of proportionality that depends on the properties of the material itself. To be more specific, \( Y \) is independent of the length or area of
the material. The units of $Y$ are $[Y] = \text{N/m}^2$, which is the SI units of Pascals $1 \text{ Pa} = \text{N/m}^2$. Comparing equations [1] and [2], we find the effective spring constant of the solid is,

$$k_{eff} = \frac{YA}{L}$$  \hspace{1cm} (3)

2. Elastic Potential Energy: the potential energy stored in a solid stretched or compressed by $\Delta L$ is,

$$U_{elastic}(\Delta L) = \frac{1}{2}k_{eff}(\Delta L)^2$$  \hspace{1cm} (4)

where $k_{eff} = \frac{YA}{L}$

3. Tensile Stress $\sigma$: defined by the force applied on a material per area of the material,

$$\sigma = \frac{F}{A}$$  \hspace{1cm} (5)

4. Strain $\epsilon$: describes the relative deformation of the material,

$$\epsilon = \frac{\Delta L}{L}$$  \hspace{1cm} (6)

5. Stress-Strain Relationship:

From equations [2], [5], and [6] we can find a linear relationship between stress and strain,

$$\sigma = Y\epsilon$$  \hspace{1cm} (7)

6. Tensile Strength $\sigma_{max}$: describes the maximum amount of stress a material can take before breaking or deformation.

7. Pressure: is defined by the force $F$ per perpendicular surface area $A$,

$$P = \frac{F}{A}.$$  \hspace{1cm} (8)

Pressure is a scalar quantity $^1$ and has units of $[P] = \text{N/m}^2 = 1 \text{ Pa}$. The unit of 1 atm = 101325 Pa is the average pressure of the Earth’s atmosphere at sea level.

8. Pressure in fluids:

(a) Pressure exists at all points in a fluid.
(b) Pressure pushes equally in all directions at a given point.

9. Fluid Statics: the pressure of a fluid at rest in a container arises from the weight of the fluid pulling down. In this case, the pressure varies only with depth $h$ of the fluid in the container,

$$p(h) = p_0 + \rho gh$$  \hspace{1cm} (9)

where $\rho$ is the density of the fluid and $p_0$ is the pressure at the top of the fluid. This means that pressure is constant along a horizontal lines (i.e. $y =$constant).

---

$^1$Very often we associate pressure to have a particular direction. However, pressure cannot have a direction since it is a scalar quantity. However, force that creates the pressure has a direction.
10. Pascal’s Principle: a pressure change occurring anywhere in a confined incompressible fluid is transmitted throughout the fluid such that the same change occurs everywhere. This means that when you press down with a piston at one point in the fluid, the pressure will change for the entire fluid, not just locally.

11. Gauge pressure $P_g$: describes the pressure above 1 atm,

$$P_g = P - 1 \text{ atm}$$

where $P$ is the total pressure at a particular point. Gauge pressure is extremely useful as it tells us by how much the total pressure is above the atmospheric pressure. The gauge pressure is what is measured when you have your blood pressure taken.

12. Buoyancy force: Archimedes principle states that when a body is fully or partially submerged in a fluid, the fluid pushes upward with a buoyant force with magnitude

$$F_b = m_{f,\text{dis}}g = \rho_f V_{f,\text{dis}}g$$

where $m_{f,\text{dis}}$ and $V_{f,\text{dis}}$ are the mass and the volume of the fluid that has been pushed out of the way by the body.

13. The apparent weight of a body: is related to its actual weight by:

$$\text{weight}_{\text{app}} = \text{weight} - F_b$$

14. Surface tension: is a boundary effect that impacts the boundary conditions. The surface tension constant $\gamma$ is defined as the increase in internal energy $\Delta U_{\text{int}}$, due to the unpaired bonds, per unit of area:

$$\gamma = \frac{\Delta U_{\text{int}}}{A}$$

where $[\gamma] = \frac{J}{m^2} = \frac{N}{m} = \frac{kg}{s^2}$

15. Surface Tension Force: The surface tension manifests as a tension force acting ON the boundary of the interface between the air+liquid:

$$F = \gamma L$$

where $L$ is typically the length around the boundary of the air-water interface.

16. Surfactants: Surfactants have polar heads and nonpolar tails. They reduce surface tension of a fluid when added to the fluid.

17. Capillary rise: The height $h$ of a liquid column is given by,

$$h = \frac{2\gamma}{\rho gr}$$

where $\rho$ is the density of the fluid, $r$ is the radius of the pipe, $\gamma$ is the liquid-air surface tension. The origins of capillary actions is the liquid’s adhesion to the container’s wall and its cohesion to molecules within the bulk (i.e. the surface tension between the water and air).
18. Laplace Pressure: The pressure jump across a spherical bubble is,

\[ P_{in} - P_{out} = \frac{2\gamma}{R} \]  

19. Ideal Fluid Model: describes fluid that is (nearly) incompressible and non-viscous with Smooth, laminar flow.

20. Continuity Equation: For steady state flow, the amount of fluid moving across a boundary must be constant in time,

\[ Q = A_1 v_1 = A_2 v_2 \]  

where \( Q \) is the volumetric flow rate, cross-sectional areas, \( A_1 \) and \( A_2 \), and speed \( v_1 \) and \( v_2 \). The expression above is a statement of mass conservation: the amount of fluid going through one cross-sectional area \( A_1 \) must be the same as the amount of fluid through \( A_2 \), \( m_{in} = m_{out} \).

21. Ideal gas: is one for which the pressure \( P \), volume \( V \), and temperature \( T \) are related by

\[ PV = nRT, \]  

where \( n \) is the number of moles of the gas present and \( R = 8.31 J/mol \cdot K \) is the gas constant. The ideal gas law can also be written as

\[ PV = NkT, \]  

where \( k = 1.38 \times 10^{-23} \) is the Boltzmann constant.
2 Selected Problems

The difficulty of each problem is rated on a scale of 1-5, with 1 being easy (novice level) and 5 being the most difficult (expert level). For context, Sapling problems generally have a difficulty level of 1-2 and offline HW problems generally have a difficulty level of 3-4. Bonus questions in lecture have a difficulty level of 4-5.

1. (2/5) A 70 kg mountain climber dangling by a rope in a crevasse stretches a 50 m long, 0.01 m-diameter rope by 0.08 m. What is Young’s modulus for the rope?

2. (3/5) A wire is stretched right to the breaking point by a 5000 N force. A longer wire made of the same material has the same diameter. Is the force that will stretch it right to the breaking point larger than, smaller than, or equal to 5000 N? Explain.

3. (1/5) The absolute pressure at a depth $h$ below the surface of the ocean is $1.5 \ P_{\text{atm}}$, where $P_{\text{atm}}$ is the atmospheric pressure. The absolute pressure at a depth $2h$ is then

(a) $2 P_{\text{atm}}$

(b) $3 P_{\text{atm}}$

(c) $4 P_{\text{atm}}$

(d) $6 P_{\text{atm}}$

(e) $9 P_{\text{atm}}$

4. (3/5) To determine an athlete’s body fat, she is weighed first in air and then again while she is completely under water. It is found that she weighs 944 N when weighed in air and 44 N when weighed underwater.

(a) What is her average density?

5. (1/5) The figure shows a pipe and gives the volume flow rate (in cm$^3$/s) and the direction of flow for all but one section. What are the volume flow rate and the direction of flow for that section?

---

*Note that some of the Sapling problems have a difficulty level of 3-4.*
6. (3/5) Floating piston

Consider a 10 kg piston floating on top of a 100 cm column of oil. The column of oil is connected to a closed container of oil on the right with height and width of 60 cm and 20 cm, respectively (See figure). The density of oil is $\rho_{\text{oil}} = 920 \text{ kg/m}^3$.

(a) How much force does the fluid exert on the end of the cylinder at point A?

(b) How much force does the fluid exert on the end of the cylinder at point B?

7. (2/5) Rank in order, from largest to smallest, the magnitudes of the forces $\vec{F}_a$, $\vec{F}_b$, and $\vec{F}_c$.

8. (2/5) The force needed to move a $l = 0.07 \text{ m}$ long wire to the right at constant velocity is $F_{\text{app}} = 5.1 \times 10^{-3} \text{ N}$. (See figure below.) Calculate the surface tension $\gamma$ of the enclosed fluid.
9. Du Noüy ring method

(3/5) A circular ring (radius \( r = 0.05 \text{ m} \)) of weight 0.02 N is used to determine the surface tension of a liquid. The plane of the ring is positioned so that it is parallel to the surface of the liquid. The ring is immersed in the liquid and then pulled upward, so a film is formed between the ring and the liquid (See figure). An upward force of \( F_{\text{app}} = 3.6 \times 10^{-2} \text{ N} \) is required to lift the ring to the point where it just breaks free of the surface. What is the surface tension of the liquid?

10. (2/5) A capillary tube has radius \( R = 0.15 \text{ mm} \). Fluid X, which has a density \( \rho_X = 1200 \text{ kg/m}^3 \), is poured into a capillary tube and the fluid rises to a height of \( h = 12.78 \text{ cm} \). What is the surface tension force per unit length constant \( \gamma_X \)?

11. (5/5) Suppose that a bubble has the shape of a long cylinder, rather than that of a sphere, with radius \( R \) and length \( L \). Determine an expression for the difference between the inside and outside pressures. Express your answer in terms of the surface tension and the radius \( R \) of the cylinder. Hint: For the cylindrical bubble, cut the cylinder into two halves by slicing along a line that is parallel to the axis of the cylinder.

12. (5/5) A cylinder of density \( \rho_0 \), total length \( L \), and cross-section area \( A \) floats in a liquid of density \( \rho_f \) with its axis perpendicular to the surface. Length \( h \) of the cylinder is submerged when the cylinder floats at rest.

(a) Show that \( h = \frac{\rho_0}{\rho_f} L \) when the cylinder is in static equilibrium.

(b) Suppose the cylinder is distance \( \Delta y \) above its equilibrium position and released. Show that the cylinder exhibits simple harmonic motion by finding an expression for the acceleration in the \( y \)-direction \( a_y \) of the cylinder. What is the angular frequency \( \omega \) of the cylinder’s oscillations?

(c) What is the “spring constant” \( k \)?

(d) If you push a floating object down and release it, it bobs up and down, and hence exhibits simple harmonic motion. Use your spring constant (or angular frequency) to show that the cylinder’s oscillation period \( T \) is:

\[
T = 2\pi \sqrt{\frac{h}{g}}
\]
13. (2/5) An air-filled balloon has a radius \( r = 20 \text{cm} \) in a room with a temperature 25°C. The balloon flies out of the open window into the street. If the temperature outside is −50°C, what will be the radius of the balloon after a long time?
3 Solutions to Selected Problem

1. **Answer: 5.76 GPa**

   The Young’s modulus is given by,
   \[ Y = \frac{\sigma}{\epsilon} = \frac{FL}{A\Delta L} \quad (21) \]

   We are given the stretch of the rope \( \Delta L = 0.08 \text{ m} \) and its original length \( L = 50 \text{ m} \). We are also given the diameter of the rope, which allows us to calculate the cross-sectional area of the rope,
   \[ A = \frac{\pi d^2}{4} = \frac{\pi (0.01 \text{ m})^2}{4} = 0.000076 \text{ m}^2 \quad (22) \]

   The force applied to the wire is just the tension in the rope, which is equal to the weight of the rock climber
   \[ F_T = mg = 700 \text{ N} \quad (23) \]

   where we used \( g = 10 \text{m/s}^2 \). Substituting this into equation \(21\) gives
   \[ Y = \frac{(700 \text{ N})(50 \text{ m})}{(0.000076 \text{ m}^2)(0.08 \text{ m})} = 5.76 \times 10^9 \text{ N/m} \quad (24) \]

2. **Answer: 5000 N**

   Wire 1 is stretched right to its breaking point with an applied force \( F_{\text{app,max}1} = 5000 \text{ N} \). Wire 2 has the same diameter as wire 1, so they have the same area, \( A_1 = A_2 = A \). This means that both wires will have the same force per area, \( \frac{F_{\text{app}}}{A} \). Since they are made of the same material, \( Y_1 = Y_2 = Y \). Although, wire 2 is longer than wire 1, \( L_2 > L_1 \), the strain of each wire \( \frac{\Delta L}{L} \) is identical in form,
   \[ \frac{\Delta L_2}{L_2} = \frac{F_{\text{app,max}2}}{AY} \quad (25) \]
   \[ \frac{\Delta L_1}{L_1} = \frac{F_{\text{app,max}1}}{AY} \quad (26) \]

   From the symmetry of these two equations, we see that \( F_{\text{app,max}2} = F_{\text{app,max}1} = 5000 \text{ N} \). In words, wire 2 breaks at the same applied force as wire 1.

3. **Answer: (a)**

   The hydrostatic pressure of a fluid varies with depth \( h \) from the surface,
   \[ P(h) = P_{\text{atm}} + \rho gh \quad (27) \]

   where \( \rho \) is the density of the fluid. We are told that pressure at a depth \( h \) is \( P_1 = 1.5P_{\text{atm}} \), and we are asked to find the pressure at a depth of \( 2h \), \( P_2 \). We can use equation \(27\) to solve for how much the pressure changes at depth \( h \),
   \[ P_1 = 1.5P_{\text{atm}} = P_{\text{atm}} + \rho gh \Rightarrow \rho gh = 0.5P_{\text{atm}} \quad (28) \]

   We can use this to solve for the pressure at \( 2h \),
   \[ P_2 = P_{\text{atm}} + \rho g(2h) = P_{\text{atm}} + 2(\rho gh) = P_{\text{atm}} + 2(0.5P_{\text{atm}}) = 2P_{\text{atm}} \quad (29) \]
4. Apparent Weight of Athlete

(a) **Answer: 1.068 g/cm³**

Below is the schematic idea of the swimmer being weighed out of water (left) and inside water (right). The scale reads the tension force $T$ holding up on the object. When the swimmer is weighed outside of water, the tension force exactly matches her weight $W = 940$ N. When she is fully submerged in water, she experiences a buoyant force $F_b = \rho_{\text{water}} V_{\text{girl}} g$.

We can solve for the density of the girl using Newton’s 2nd Law (where $a_y = 0$ since she is at rest). The plan will be to first solve for her volume, since we could find her mass from her weight.

$$F_{\text{net,y}} = F_T + F_b - mg = 0 \Rightarrow F_T + \rho_{\text{water}} V_{\text{girl}} g - mg = 0 \Rightarrow V_{\text{girl}} = \frac{mg - F_T}{\rho_{\text{water}} g}$$

Since the scale reads 44 N when she is underwater, $F_T = 44$ N, $mg$ is 940N, and we know the density of water and $g$. Giving us a volume of the girl $= 0.091$ m$^3$. With this, we find that her average density is,

$$\rho_{\text{girl}} = \frac{944 \text{ N}}{0.091 \text{m}^3(9.8 \text{ m/s}^2)} = 1060.0 \text{ kg/m}^3$$

5. **Answer: 13 cm³ out of the pipe**

The continuity equation states that the conservation of the mass of the fluid $m_{in} = m_{out}$ that for ideal incompressible fluid becomes the conservation of volume (dividing by $\rho_f$):

$$V_{in} = V_{out}.$$
If the unknown volume flow rate is $V_X$ and we assume that it flows into the pipe, then the equation above implies

$$4 + V_X + 8 + 4 + 5 = 6 + 2 \implies V_X = -13 \text{cm}^3/\text{s} \quad (35)$$

The minus sign indicates that our assumption is incorrect and the water in fact flows out of the pipe.

6. Floating Piston

(a) Answer: $P_A = 1.85 \times 10^5 \text{ Pa}$

According to the hydrostatic equation, the pressure within a static fluid only varies with height. Therefore, the pressure at point $A$ is,

$$P_A = P_{top} + \rho g h_{top \rightarrow A} \quad (36)$$

where $h_{top \rightarrow A} = 0.70 \text{ m}$. Note that this assumes the pipe inserts halfway into the right compartment of fluid. The pressure at the top includes the atmospheric pressure plus the additional pressure created by the weight of the 10 kg piston, $F_{g,piston} = 98 \text{ N}$. Therefore, at the pressure at the top of the fluid column is,

$$P_{top} = P_{atm} + \frac{F_{g,piston}}{A} = 1.01 \times 10^5 \text{ Pa} + \frac{98 \text{ N}}{0.00125 \text{ m}^2} = 1.78 \times 10^5 \text{ Pa} \quad (37)$$

Plugging this into equation (36) we can find the pressure at point A

$$P_A = P_{top} + \rho g h_{top \rightarrow A} = 1.78 \times 10^5 \text{ Pa} + (920) \frac{\text{kg}}{\text{m}^3}(9.8 \text{ m/s}^2)(0.7 \text{ m}) = 1.85 \times 10^5 \text{ Pa} \quad (38)$$

(b) Answer:

According to the hydrostatic equation, the pressure within a static fluid only varies with height. Therefore, the pressure at point $B$ is,

$$P_B = P_A + \rho g h_{A \rightarrow B} = 1.85 \times 10^5 \text{ Pa} + (920) \frac{\text{kg}}{\text{m}^3}(9.8 \text{ m/s}^2)(0.6 \text{ m}) = 1.91 \times 10^5 \text{ Pa} \quad (39)$$

where we used $h_{A \rightarrow B} = 0.6 \text{ m}$.

7. Answer: $F_b > F_a = F_c$

In each of the three cases, an applied force is applied on the left to sustain the weight(s) on the right. According to Pascal’s Principle, when there is an increase in pressure at any point in a confined fluid, there is an equal increase at every other point in the container. Consequently, the pressure will be the same along horizontal lines. This means that the pressure on the left $P_L$ right under the piston due to the force on the left is the same as the pressure on the right $P_R$ right under the piston. Recalling that pressure has a magnitude of force per cross-sectional area, we have

$$P_L = \frac{F_L}{A_L} \quad (40)$$

$$P_R = \frac{F_R}{A_R} \quad (41)$$

$$P_L = P_R \implies F_L = \frac{A_L}{A_R} F_R \quad (42)$$

This applies only when the fluid is in static equilibrium.
For all three cases, the right and left cross-sectional areas on the sides $A_L$ and $A_R$ identical, which $A_L < A_R$. This means that the largest force $F_L$ that balances the weight will have the largest $F_R$. For case (a) and (c), $F_R = F_g = (600 \text{ kg})(10 \text{ m/s}^2) = 6000 \text{ N}$. For case (b), $F_R = F_g = (1000 \text{ kg})(10 \text{ m/s}^2) = 10000 \text{ N}$. Therefore, $F_b > F_a = F_c$.

8. Answer: $\gamma = 0.0364 \text{ kg/s}^2$

The applied force required to move a wire of length $l$ connected to a soap film at constant velocity must counter the force of surface tension to the left,

$$F_{app} = F_T = 2\gamma l$$

If $F_{app} = 5.1 \times 10^{-3} \text{ N}$, then $\gamma = 0.0364 \text{ kg/s}^2$.

9. Answer: $\gamma = 0.0255 \text{ N/m}$

The forces acting on the ring include the downward surface tension force $F_T = -2\gamma(2\pi r)\hat{y}$, the force of gravity, $F_g = -mg\hat{y} = 0.02 \text{ N}\hat{y}$, and the upward applied force, $F_{app} = 0.036 \text{ N}\hat{y}$.

Right before the ring breaks the surface, it is in static equilibrium,

$$F_{net,y} = F_{app} - F_T - F_g = 0 \quad \Rightarrow \quad F_T = 2\gamma(2\pi (0.05 \text{ m})) = F_{app} - F_g = 0.016 \text{ N}$$

$$\Rightarrow \quad \gamma = \frac{0.016 \text{ N}}{2(2\pi (0.05 \text{ m}))} = 0.0255 \text{ N/m}$$

10. Answer: $\gamma_X = 0.115$

The equation for capillary rise is,

$$h = \frac{2\gamma_X}{\rho_X g \sqrt{r}}$$

where we can solve for the surface tension force per unit length parameter,

$$\gamma_X = \frac{\rho_X g h r}{2} = \frac{(1200) \text{ kg/m}^3(10 \text{ m/s}^2)(0.00015 \text{ m})(0.1278 \text{ m})}{2} = 0.115 \text{ kg/s}^2$$

Remark: The corresponding density $\rho_X = 1200$ and surface tension $\gamma_X = 0.115$ correspond to that for a 26% NaCl solution at standard temperature.

11. Answer: $P_{in} - P_{out} = \frac{\gamma}{r}$

Following the suggestion of the problem, we divide the cylinder down its center with a rectangular sheet of area $A = 2RL$ (See Figures). Since the bubble is in static equilibrium, We can then perform a force balance on the cross-sectional rectangular area. The force on the outside of the bubble relates to the pressure outside the bubble, $F_{out} = P_{out}A = P_{out}(2RL)$. The force on the top relates to the pressure inside of the bubble, $F_{in} = P_{in}A = P_{in}(2RL)$. The force of

---

4The factor of 2 comes from the fact that there are two interfaces between air and liquid, as we had with the problem before.
Surface tension acts downward **along the length of the rectangle**, as an elastic balloon does to contain the inward pressure, $F_T = \gamma (2L)$. Putting this together we have,

$$F_{\text{net},y} = F_{\text{in}} - F_{\text{out}} - F_T = 0 \Rightarrow P_{\text{in}}(2RL) - P_{\text{out}}(2RL) - F_T(2L) = 0$$

$$\Rightarrow P_{\text{in}} - P_{\text{out}} = \frac{\gamma}{R}$$

*Expert Remark:* An easier, more mathematical way to derive this involves using the definition of surface tension. The surface tension measures the amount of work (and hence energy) to increase the amount of exposed surface. By work, we mean the amount of work required to change the volume of the bubble at fixed pressure difference, $dW = (\Delta p) dV$

$$\gamma = \frac{dW}{dA} = \frac{(\Delta p)dV}{dA}$$

The volume and surface area around the cylinder are, $V = \pi R^2 L$ and $A = 2\pi RL$, respectively. We can use the chain rule to express $\frac{dV}{dA}$ as $\frac{dV/dR}{dA/dR}$, which gives us,

$$\gamma = \Delta p \frac{dV}{dA} = \Delta p \frac{dV/dR}{dA/dR} = \Delta p \frac{2\pi RL}{2\pi L} \Rightarrow \Delta p = \frac{\gamma}{R}$$

12. Bouncing Buoy

(a) *Answer: See Below*

The net force on the cylinder must be zero in static equilibrium. The amount of fluid volume displaced by the cylinder, with cross-section area $A$, is $V_{\text{dis}} = Ah$, and the mass of the object can be written as $m_0 = \rho_0 V_0 = \rho_0 AL$. Putting this together, we have

$$F_{\text{net},y} = F_B - F_g = 0 \Rightarrow F_B = F_g \Rightarrow \rho_f (Ah)g = \rho_0 (AL)g \Rightarrow h = \frac{\rho_0}{\rho_f} L$$

(b) *Answer: $a_y = -\frac{g}{h} y$*

At this point it may seem like our expression for the acceleration cannot be simplified any further. However, it can be by using the expression we found for $h$ in part (a), $h = \frac{\rho_0 L}{\rho_f}$,

$$a_y = \left( \frac{\rho_f}{\rho_0 L} \frac{\rho_0 L}{\rho_f} - 1 \right) g - \frac{\rho_f g}{\rho_0 L}y = -\frac{\rho_f g}{\rho_0 L}y = -\frac{g}{h} y$$
When the cylinder is pulled up from its equilibrium position by $y$, the amount submerged is $V_{dis} = A(h - y)$. We can then calculate the net force on the object, which will now be nonzero

$$F_{net,y} = F_B - F_g = ma_y \Rightarrow \rho_f \left( A(h - y) \right) g - \rho_0 g (AL) = \rho_0 (AL) a_y$$

$$\Rightarrow \left( \frac{\rho_f}{\rho_0 L} h - 1 \right) g - \frac{\rho_f g}{\rho_0 L} y = a_y \quad (51)$$

Since the acceleration is of the form,

$$a_y = -\left( \frac{g}{h} \right) y \Rightarrow a_y = -\omega^2 y$$

we know that the cylinder will exhibit simple harmonic motion with an angular frequency of $\omega = \sqrt{\frac{g}{h}}$.

(c) **Answer:** $k = \rho_0 A L g$

We know that angular frequency squared relates to the effective spring constant and mass of the oscillating cylinder, $\omega^2 = \frac{k}{m}$. From our expression in part (b), we see that $\omega^2 = \left( \frac{g}{h} \right)$. Relating the two allows us to solve for the effective spring constant,

$$k = \frac{mg}{h} = \frac{\rho_0 A L g}{h} \quad (54)$$

which has the appropriate units of the spring constant $[k] = \text{kg/s}^2$.

(d) **Answer:** $T = 2\pi \sqrt{\frac{k}{g}}$

The relationship between the period of oscillations and frequency are,

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{k}{g}} \quad (55)$$

13. **Answer:** 18.3 cm

According to the ideal gas law

$$PV = \nu RT.$$  \hspace{1cm} (56)

Throughout the problem the gas is trapped in the balloon, so the number of moles doesn’t change. The same goes for the pressure that remains atmospheric. Then for the balloon in the room and on the street we can write

$$PV_{room} = \nu RT_{room}$$
$$PV_{street} = \nu RT_{street} \quad (57)$$

By dividing these equations and recalling that the ballon is a sphere to a good approximation,

$$\frac{R_{street}}{R_{room}} = \frac{3}{3} \frac{V_{street}}{V_{room}} = \frac{3}{3} \frac{T_{street}}{T_{room}}.$$  \hspace{1cm} (58)

Solving for the radius we get

$$R_{street} = R_{room} \sqrt[3]{\frac{T_{street}}{T_{room}}} = 20 \text{ cm} \times \sqrt[3]{\frac{(273 - 50) \text{ K}}{(273 + 20) \text{ K}}} = 18.3 \text{ cm.} \quad (59)$$