Physics 289

Lecture III → Gas/Fluid Dynamics of Viscous Disk

→ Disk Transport Model

- streaming $V_s$

Recall: derived bulk equations

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma v_r) = 0$$

(continuity)

$$\frac{\partial}{\partial t} (r^2 \Sigma v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r v_r \Sigma r v_\theta) = 0$$

(regular momentum)

$$= \frac{1}{r} \frac{\partial}{\partial r} (r^2 T_{\phi \phi})$$

→ viscous stress

$$T_{\phi \phi} = \Sigma r v_\theta \frac{\partial \Sigma}{\partial r}$$

Viscous stress $\sim D^2$, rotation shear $\sim$ dynamics in $V$.

and exploit $v_r \ll C_s, v_\theta \rightarrow$ ordering $\Rightarrow$ time scale separation
\[ \nu_r = \frac{\rho_r}{\frac{1}{2} \left( \frac{r}{\sum \nu_r} \right)} \]

\[ \nu_r \approx \frac{3}{2} \left( \frac{r}{\sum \nu_r^{1/2}} \right) \]

\[ \sum \nu_r^{1/2} \approx \left( \frac{r}{\sum \nu_r^{1/2}} \right) \]

and:

\[ \frac{\partial \Sigma}{\partial t} = -\frac{3}{2} \frac{\partial}{\partial r} \left[ \frac{1}{\nu_r^{1/2}} \left( \frac{r}{\sum \nu_r^{1/2}} \right) \right] \]

*an important equation*

steady state solutions

Follows:

\[ \dot{M} = -2\pi r \nu_r \Sigma \]  

(continuity and \( \nu_r \))

acceleration \( \gg 0 \), as \( \nu_r \ll \Sigma \)

and
\[ dL_{\text{out}} = \int_{\text{tangents}} 2\pi r dr \frac{d}{dr} \left( \frac{r^2 T_{\phi}}{r \, dv} \right) \]

\[ = -2\pi \sum r^3 \omega' \]

(Inner spins up, outer spins down)

\[ \frac{dL_{\text{out}}}{dt} \]

so can integrate angular momentum equation:

\[ \int^{\infty}_{2\pi r dr} \left[ \frac{d}{dt} \left( \sum r \omega' \right) + \frac{1}{r \, dv} \left( \sum r \frac{\partial \omega'}{\partial r} \right) \right] \]

\[ = \frac{1}{r \, dv} \left( \frac{d}{dt} \left( \sum r^2 \omega' \right) \right) \]

\[ \dot{J} = \left[ 2\pi r \omega' \sum \frac{r^3 \omega'}{r \, dv} \right] \]

Note change of disk angular momentum:

\[ J = \left[ 2\pi r \omega \sum \frac{r^2 \omega}{r \, dv} \right] \]

\[ -2\pi r^2 T_{\phi} \]
\[ J = -M_0 \rho_i r_i^2 - 2\pi \sum r_i^3 \bar{R}_i \]  

Flux associated with accretion process (\( - \))

Flux associated with viscous torque (\( + \))

\[ \bar{D} < 0 \]

(1), (2) steady state solutions.

Frequent assumption: "No torque"

\[ \frac{\partial}{\partial r} \bar{D} |_{r = 0} = 0 \]

inner boundary condition is critical

- Why? Boundary layer where disk connects to central object.

Detailed microphysics!
So, in no torque assumption:
\[ J = -\, \dot{M} \, \Sigma \, R^2 \, \rho_{in} \left[ 1 \right] \rightarrow a \]

\[ \frac{1}{J} \, 1 = \dot{M} \left( \sigma M \rho_{in} \right)^{1/2} \]

β:
\[ \left\{ \begin{array}{c}
3 \pi v \Sigma = \dot{M} \left( 1 - \frac{v_{in}}{v} \right)^{1/2} \\
\nu \gg \nu_{li} \\
\dot{M} = \beta \pi v \Sigma
\end{array} \right. \]

\[ \Rightarrow \text{finishes basic disk solutions.} \]
- **Energetics**

  → Astronomer: "How predict/calculate luminosity of disk?"

  → Physicist: "What happens to energy/power excited against viscous stresses?"

  **Answer**: Heating!

N.B. From N-S: \[ \frac{d}{dt} \int_0^\infty v^2 \, dt = \int_0^\infty \gamma (\theta V)^2 \, dv \]

or \[ \int_0^\infty \frac{d}{dt} v^2 \, dt = \int_0^\infty v (\theta V)^2 \, dv \]

which is the mechanical energy decay

so \[ \Delta \int E_{\text{Thermal}} = + \int_0^\infty \gamma (\theta V)^2 \, dv \]

thermal energy gain: heating
Now, consider annulus of disk.

\[ \text{annulus} \]

\[ r - \Delta r \quad \text{\&} \quad r + \Delta r \]

If recall:

\[ G(r) = -2\pi r^2 \tau \]

outward flux, due viscous stress, of angular momentum thru r surface.

Then net torque on annulus \( \Delta r \) is:

\[ \tau = G(r - \Delta r) - G(r + \Delta r) \]

\[ \frac{\Delta\tau}{\Delta r} \quad \text{or} \quad \frac{\partial G}{\partial r} \]
so power produced by torque:

\[ P = \alpha E = \omega \tau \]

\[ \frac{P}{\Delta r} = -\frac{\alpha}{2} \frac{2E}{\Delta r} \]

\[ \text{power per interval} \]

\[ = -\left( \frac{1}{2\pi} \left( \frac{\partial}{\partial r} (5\Omega) - \frac{\partial}{\partial r} \right) \right) \]

\( \text{Key:} \)

1 = energy transport

2 = no integrated contribution except at boundary

2 = heating via energy dissipation.

\[ 0(r) = \text{power dissipated per area (watt/m}^2\text{)} \]

by viscosity = heating
\[ D(r) = -\frac{G \rho l^{'}}{2 \pi r} \quad \text{(origin to disk)} \]

Total power emitted by disk in steady state:

\[ L_{\text{disk}} = \int_{r_{1}}^{r_{\infty}} 2\pi r \, D(r) \, dr \]

Now:

\[ G(r) = -2\pi r^2 T_{\theta \phi} \]

\[ T_{\theta \phi} = \sum_{r} r \, d \theta / dr \]

\[ 0 = \rho \sum \left[ \theta \frac{d \theta}{dr} \right]^2 \]

\[ D(r) = \rho \sum \left[ \rho \frac{d l^{'}}{dr} \right]^2 \quad \text{- mixing form} \]

\[ \rho (\theta \nu)^2 \quad \text{with respect to solid body} \]
\[ L_{\text{disk}} = \int_{r_0}^{\infty} 2\pi r [\gamma \sum (\pi r')^2] \, dr \]

but \[ 2\pi r \sum = \dot{M} \left( 1 - \sqrt{\frac{a}{r}} \right) \]

\[ \Rightarrow \]

\[ L_{\text{disk}} = \sqrt{\frac{3GMM}{r_0}} \left( a - \left( \frac{r_0}{r} \right)^{1/2} \right) \]

\[ \Rightarrow \frac{\dot{L}}{\sqrt{A}} \frac{\dot{M}}{M} = \text{from disk} \]

- Potential energy lost by accretion

\[ \dot{P}E \sim \frac{GM^2 M}{\sqrt{r_0}} \]

\[ L(\text{dot}) = \frac{1}{2} \dot{P}E \]

Rest goes to maintain \( J(\text{dot}) \pi \).