# CS30 (Discrete Math in CS), Summer 2021 : Lecture 24 + 25

Topic: Graphs: Matchings and Hall's Theorem

*Disclaimer: These notes have not gone through scrutiny and in all probability contain errors.*
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- **Matchings.** A ***matching*** $M$ in a graph is a subset of edges $M \subseteq E$ such that for any $e, e' \in M$, $e \cap e' = \emptyset$. That is, $M$ is a collection of edges which do not share end points. A vertex $v \in V$ participates in the matching $M$ if there is an edge in $M$ which is incident to $v$.

  These are *fundamental objects* and have numerous applications. For instance, in economics, where the bipartite graph contains agents on one side and items on the other, where the edges represent desirable items, and each agent has only a demand of one item, then a matching corresponds to an allocation of desirable items to these agents.

  A matching is a ***perfect matching*** if every vertex of $V$ appears in some edge of the matching.

- **Matchings in Bipartite Graphs.** In this course, we look at matchings in bipartite graphs. To this end, fix a bipartite graph $G = (V, E)$ where $V$ has been partitioned in to $L \cup R$. We say that a matching $M \subseteq E$ is an $L$-matching if all vertices in $L$ participate in $M$. Similarly, a matching $M$ is an $R$-matching if all vertices in $R$ participate in $M$. A bipartite graph has a perfect matching if and only if it has an $L$-matching and an $R$-matching.

  Given a graph, how can we tell whether or not there is an $L$-matching (likewise $R$-matching)? Today, we are going to state an *amazing* theorem (called Hall's theorem) which gives the necessary and sufficient conditions for a bipartite graph to have an $L$-matching. Then, we look at some *applications* of this theorem. Next class, we will prove this remarkable theorem. This may be the **deepest theorem** you learn in this course.

  Before, we state the theorem, let us recall some notions. The ***neighborhood*** of a vertex $v$ in $G$ is the set $N_G(v) := \{u \in V : (u, v) \in E\}$. Note that when $G$ is bipartite, and if $v \in L$, then $N_G(v) \subseteq R$. And vice-versa. Next, we *generalize* the definition of neighborhood to *subsets of vertices*. Given any subset $S \subseteq L$, we define

  $$N_G(S) := \bigcup_{v \in S} N_G(v)$$

  That is, we take the union of all the neighborhoods of vertices in $S$. In English, $N_G(S)$ is the set of vertices in $R$ which have at least one neighboring vertex in $S$. Figure 1 shows three examples of subsets and their neighborhoods in a given graph

  Now we are ready to state Hall's theorem.

  **Theorem 1** (Hall's Theorem)**.** Let $G = (V, E)$ be a bipartite graph with $V = L \cup R$. Then, $G$ has an $L$-matching if and only if

  $$\text{For every subset } S \subseteq L, \ |N_G(S)| \geq |S|$$

  **Corollary 1.** A bipartite graph $G = (L \cup R, E)$ has a *perfect matching* if and only for any $S \subseteq L$ or $S \subseteq R$ we have $|N_G(S)| \geq |S|$.

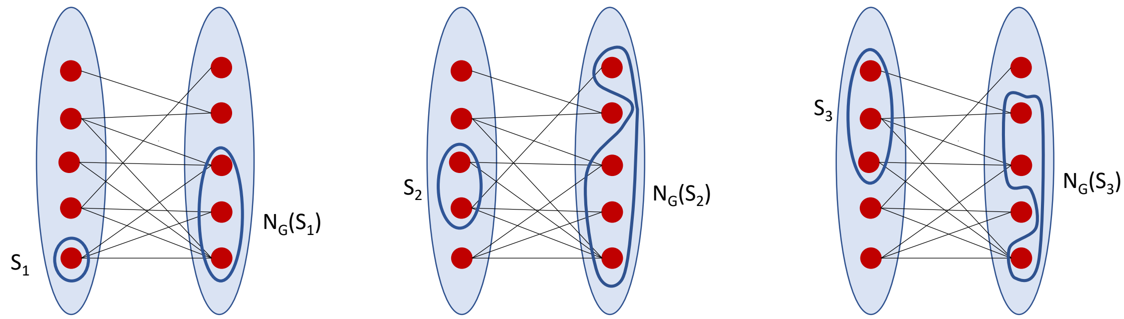


Figure 1: Examples of $S$ and $N_G(S)$ in a bipartite graph.

Going back to Figure 1, Hall's theorem says if $G$ doesn't have an $L$-matching, then there must be some subset $S \subseteq L$ such that $|N_G(S)| < |S|$. As it happens, this example has a perfect matching, and so every subset $S \subseteq L$ must have $|N_G(S)| \geq |S|$.

**Remark:** *Going back to the Prover-Verifier mode of thinking, imagine you have a bipartite graph $G = (L \cup R, E)$, and you wish to know if there is an $L$-matching or not. The Prover is all powerful, and can easily find the answer out. But you also need a* ***certificate*** *of whether what they are claiming is correct or not.*

*In case $G$ has an $L$-matching, this certificate is easy — Prover just shows the $L$-matching. As Verifiers, we check if all purported edges are indeed present and also check if they don't intersect, etc.*

*If $G$ does* ***not*** *have an $L$-matching, then the Prover resorts to Hall's theorem. They know that there* ***must exist*** *some subset $S \subseteq L$ such that $|N_G(S)| < |S|$ (if not, then Hall's condition holds, and the graph has an $L$-matching). And this subset $S$ is what they send. And we, as verifiers, figure out $N_G(S)$, see that $|N_G(S)| < |S|$, and we are now convinced $G$ cannot have an $L$-matching — if it did, then all vertices of $S$ would have to be matched to $|S|$ many distinct vertices in $N_G(S)$, but $|N_G(S)| < |S|$.*

- **Applications of Hall's Theorem** We show two applications and in the UGP we explore a few more.
  - *Left-dominant bipartite graphs.* A bipartite graph $G = (L \cup R, E)$ is *left dominant* if $\deg_G(x) \geq \deg_G(y)$ for any $x \in L$ and any $y \in R$. The Hall's theorem shows that any left-dominant graph with no isolated vertices has an $L$-matching.

    *Proof.* By Hall's theorem, it suffices to show that for any subset $S \subseteq L$, $|N_G(S)| \geq |S|$. To this end, fix a subset $S$.

    Let $D_{min} := \min_{x \in S} \deg_G(x)$ and $D_{max} := \max_{y \in N_G(S)} \deg_G(y)$. $G$ being left-dominant means $D_{min} \geq D_{max}$. No isolated vertices implies $D_{min}, D_{max} \neq 0$.

    Now consider the graph $H$ induced by $(S \cup N_G(S))$; let $H = (S \cup N_G(S), E_S)$. Note that $\deg_H(x) = \deg_G(x)$ for all $x \in S$, and $\deg_H(y) \leq \deg_G(y)$ for all $v \in N_G(S)$. The latter holds since we only delete vertices and edges; the former holds because all neighbors of $x \in S$ are present in $H$.

Next note (from the drill)

(a) $|E_S| = \sum_{u \in S} \deg_H(v) = \sum_{u \in S} \deg_G(u) \geq D_{min} \cdot |S|$, and

(b) $|E_S| = \sum_{w \in N_G(S)} \deg_H(w) \leq \sum_{w \in N_G(S)} \deg_G(w) \leq D_{max} \cdot |N_G(S)|$.

Thus, we get,

$$D_{min} \cdot |S| \leq |E_S| \leq D_{max} \cdot |N_G(S)|$$

implying $|N_G(S)| \geq |S|$ since $D_{min}, D_{max} \neq 0$. □

**Exercise:** *Prove that a* regular *bipartite graph always has a perfect matching.*

– *Completeing Latin Rectangles to Latin Squares.*

A *Latin rectangle* is an $r \times n$ matrix with $r \leq n$. Each entry of the matrix has numbers from an *alphabet* $\{a_1, a_2, \ldots, a_n\}$. Think of these as colors – more vibrant that way! The constraint is that any row and any column has *no repeating entry*. So, if we go up a column or left-to-right a row, no color is repeated.

So, for example, the following are examples of Latin rectangles; one is a $2 \times 5$ and the other is a $3 \times 5$.

| | | | | |
|---|---|---|---|---|
| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
| $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_1$ |

| | | | | |
|---|---|---|---|---|
| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
| $a_3$ | $a_1$ | $a_4$ | $a_5$ | $a_2$ |
| $a_2$ | $a_5$ | $a_1$ | $a_3$ | $a_4$ |

An $n \times n$ Latin rectangle is called a Latin square. A *completion* of an $r \times n$ Latin rectangle is an $n \times n$ Latin square whose first $r$ rows is the Latin rectangle. The question is:

*Can every Latin rectangle be completed?*

And the answer is:

**Theorem 2.** Every Latin rectangle can be completed.

*Proof.* Let us fix an $r \times n$ Latin rectangle $T$. Now, we show how to construct an $(r+1) \times n$ Latin rectangle whose first $r$ rows are the rows of $T$. We can then repeat this till we get our desired Latin square.

We do so by using Hall's theorem! Pause here for a moment. There are no graphs mentioned. And yet, Hall's theorem? The main a ha! moment is to construct a bipartite graph using $T$. We do so as follows.

We construct a bipartite graph $G = (L \cup R, E)$. $L$ is the set of colors $\{a_1, a_2, \ldots, a_n\}$. $R$ is the set of positions of the $(r+1)$th row, given by $\{1, 2, \ldots, n\}$. We have an edge $(a_i, j)$ in $E$ if the color $a_i$ ***does not*** appear in the $j$th column of $T$. That is, the color $a_i$ is a feasible candidate to be put in the $j$th column of the $(r+1)$th row. This completes the description of the graph. As an illustration, for the $3 \times 5$ table shown above, we would have the graph as in Figure 2.

Now observe: if $G$ has a $L$-matching, then we can fill the $(r+1)$th row. Indeed, if the matching has the edge $(a_i, j)$ we put the color $a_i$ on the $j$th column of the $(r+1)$th row.

To show that $G$ has a perfect matching, we show that $G$ is a left-dominant graph, that is, every $x \in L$, every $y \in R$ satisfies $\deg_G(x) \geq \deg_G(y)$.

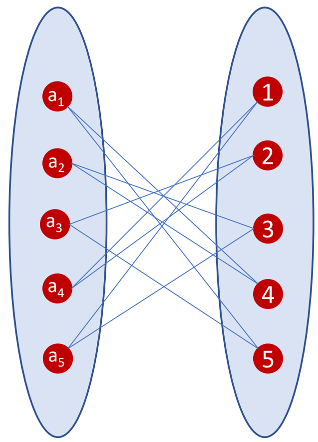


Figure 2: Construction of graph from Latin rectangle

Fix a vertex $x \in L$. What is $\deg_G(x)$? For each column of the $(r+1)$th row, exactly $r$ colors are disallowed and so $(n-r)$ colors are allowed. Thus, $\deg_G(x) = n - r$.

Now fix a vertex $y \in R$. What is $\deg_G(y)$? This is the number of columns of the $(r+1)$th row in which the number $y$ can be put. This is precisely the columns in which $y$ *doesn't* appear. But $y$ appears in $r$ different columns, and thus the number of columns free for $y$ is also $(n-r)$. Thus, not only is $G$ left-dominant, but rather it is a ***regular*** graph; all degrees are equal.

Therefore, $G$ has a $L$-matching. And thus, one can add an $(r+1)$th row to this Latin rectangle. And go on like this till one gets a Latin square.

Again, we illustrate it to get the $4$th row for the $3 \times 5$ rectangle shown in the previous page. Recall, Figure 2 was the corresponding bipartite graph. We see (as we should) that it is regular, and thus it contains an $L$-matching. Indeed, one of the matchings is shown below in Figure 3 below.

Given this matching, one sees that the $4$th row as being $(a_5, a_4, a_2, a_1, a_3)$ (because $1$ matched to $a_5$, 2 matched to $a_4$ and so on), which when slapped onto the rectangle gives us

| | | | | |
|---|---|---|---|---|
| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
| $a_3$ | $a_1$ | $a_4$ | $a_5$ | $a_2$ |
| $a_2$ | $a_5$ | $a_1$ | $a_3$ | $a_4$ |
| $a_5$ | $a_4$ | $a_2$ | $a_1$ | $a_3$ |

□

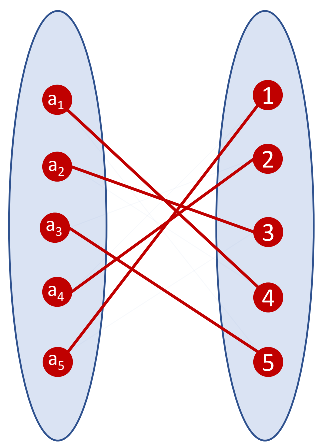


Figure 3: A matching in the graph from Latin rectangle