Math 55a Homework 8

Due Wednesday October 28, 2020.

Material covered: Finitely generated abelian groups; group actions, orbits and counting; finite subgroups of SO(3). (Artin §14.4,14.5,14.7, §6.7-6.12)

(Note: Several of the problems below review material previously covered, rather than this week's new material.)

0. Sometime over the weekend of October 24-25, please complete the week 8 survey (in Canvas).

1. Let G be a finite abelian group.

(a) For each prime p, show that the set of elements of order a power of p,

$$G_p := \{g \in G \mid g^{p^n} = e \text{ for some } n\},$$

is a subgroup of G.

(b) Show that there is a natural isomorphism $G \cong \prod_p G_p$. (Do not use the classification theorem)

(Hint: first construct a homomorphism $\varphi : \prod_p G_p \to G$ whose restriction to each factor is the inclusion $G_p \hookrightarrow G$. Next, for each prime p, write $|G| = p^k\, m$ with $p \not| m$, let r be such that $m|r$ and $r \equiv 1 \mod p^k$, and consider the homomorphism $g \mapsto g^r$ from G to itself. Show that the image of this homomorphism is G_p, and use this to construct an inverse to φ.)

2. Prove that for any pair of positive integers a, b, the group $\mathbb{Z}/a \times \mathbb{Z}/b$ is isomorphic to $\mathbb{Z}/\mathrm{lcm}(a, b) \times \mathbb{Z}/\gcd(a, b)$. Using this, prove any finite product of finite cyclic groups is isomorphic to a product of the form $\mathbb{Z}/a_1 \times \cdots \times \mathbb{Z}/a_n$, where $a_1|a_2|\cdots|a_n$. (In fact, to a *unique* such product – but you aren't required to prove uniqueness).

3. Show that, if an element of $GL(2, \mathbb{Z})$ (the group of invertible 2×2 matrices with integer coefficients) has finite order n, then $n \in \{1, 2, 3, 4, 6\}$.

(Hint: view the given element as a linear operator on a 2-dimensional complex vector space. What can you say about its eigenvalues, and about its trace?)

4. A *lattice* in the Euclidean plane $\mathbb{R}^2$ is an additive subgroup $\Lambda = \mathbb{Z}u \oplus \mathbb{Z}v \subset \mathbb{R}^2$ generated by two linearly independent vectors. Let $\Lambda \subset \mathbb{R}^2$ be a lattice, and let $G \subset O(2)$ be the subgroup of all orthogonal transformations (rotations and/or reflections) of $\mathbb{R}^2$ which map Λ to itself.

Show that every element of G has finite order (hint: Λ has finitely many shortest vectors), and use the result of the previous problem to show that G must be isomorphic to either $\mathbb{Z}/n$ or the dihedral group D_n, for some $n \in \{2, 4, 6\}$. (Optional: which of these possibilities actually occur?)

5. (a) Show that any group of order 6 is isomorphic to either $\mathbb{Z}/6$ or the symmetric group S_3.

(b) Classify all groups of order 8 up to isomorphism.

Hint for both parts: If $ab \neq ba$, then one of a, b, and ab must have order > 2. (Why?) Moreover, a subgroup $H \subset G$ with $|G/H| = 2$ must be normal. (Why?)

6. Let G be a group, not necessarily finite, and let $H \subset G$ be a subgroup of finite index, that is, such that there are finitely many left cosets of H in G. Prove that the number of right cosets is equal to the number of left cosets (so that we can define the index of H in G unambiguously). (Hint: find an operation on G which maps left cosets to right cosets).

7. Let $G \subset S_n$ be any subgroup of the symmetric group. The action of G on $\{1, \ldots, n\}$ is said to be *twice transitive* if G acts transitively on ordered pairs of distinct elements, i.e. for every $i, i', j, j' \in \{1, \ldots, n\}$ with $i \neq i'$ and $j \neq j'$, there exists $\sigma \in G$ such that $\sigma(i) = j$ and $\sigma(i') = j'$. Show that if the action of G on $\{1, \ldots, n\}$ is twice transitive and G contains a transposition then $G = S_n$.

8. Let V be a 2-dimensional vector space over the field $\mathbb{F}_p$, and $G = GL(V)$ the group of automorphisms of V (i.e., $G = GL_2(\mathbb{F}_p)$, the group of 2×2 invertible matrices with entries in $\mathbb{F}_p$).

(a) Show that V has exactly $p + 1$ 1-dimensional subspaces.

(b) Given this, we have a homomorphism $\phi : G \to S_{p+1}$, since every automorphism of V must permute its 1-dimensional subspaces. Describe the kernel and the image of ϕ for $p = 2$ and $p = 3$.

9. An element of the symmetric group S_n is called a k-cycle if it permutes k elements cyclically and fixes the remaining $n - k$. How many k-cycles are there in S_n?

10. Let G be a group of order p^n with p prime, and suppose G acts on a finite set S. Prove that if the cardinality of S is not divisible by p, then there must be an element $s \in S$ fixed by every $g \in G$; that is, an element whose stabilizer is all of G.

11. How many different bracelets can you make with 4 white beads and 4 black beads? (Hint: use Burnside's formula!)

12. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?