From Reconnection to Relaxation: A Pedagogical Tale of Two Taylors

or: The Physics Assumptions Behind the Color VG

P.H. Diamond

W.C.I. Center for Fusion Theory, N.F.R.I., RoK
and
C.M.T.F.O., U.C.S.D.

This talk focuses on:

- what is the connection between local reconnection and global relaxation?

- how do highly localized reconnection processes, for large Rm, Re, produce global self-organization and structure formation?
We attempt to:

- describe both magnetic fields and flows with similar concepts
- connect and relate to talks by H. Ji, D. Hughes, H. Li, O.D. Gurcan...
- describe self-organization principles
Outline

i.) Preamble: → From Reconnection to Relaxation and Self-Organization
   → What ‘Self-Organization’ means
   → Why Principles are important
   → Examples of turbulent self-organization
   → Preview

ii.) Focus I: Relaxation in R.F.P. (J.B. Taylor)
   → RFP relaxation, pre-Taylor
   → Taylor Theory  - Summary
      - Physics of helicity constraint + hypothesis
      - Outcome and Shortcomings
   → Dynamics → Mean Field Theory  - Theoretical Perspective
      - Pinch’s Perspective
      - Some open issues
   → Lessons Learned and Unanswered Questions
iii.) Focus II: PV Transport and Homogenization (G.I. Taylor)

→ Shear Flow Formation by (Flux-Driven) Wave Turbulence

→ PV and its meaning; representative systems

→ **Original Idea:** G.I. Taylor, Phil. Trans, 1915, ‘Eddy Motion in the Atmosphere’
  - Eddy Viscosity, PV Transport and Flow Formation
  - Application: Rayleigh from PV perspective

→ Relaxation: PV Homogenization (Prandtl, Batchelor, Rhines, Young)
  - Basic Ideas
  - Proof of PV Homogenization
  - Time Scales
  - Relation to Flux Expulsion
  - Relation to Minimum Enstrophy states
Outline

→ Does PV Homogenize in Zonal Flows?
  - Physical model and Ideas
  - PV Transport and Potential Enstrophy Balance
  - Momentum Theorems (Charney-Drazin) and Incomplete Homogenization
  - RMP Effects
  - $B_0$ Effects
  - Lessons Learned and Unanswered Questions

→ Discussion and General Lessons Learned
I.) Preamble

→ From Reconnection to Relaxation

- Usually envision as localized event involving irreversibility, dissipation etc. at a singularity

- ⬤ - how describe **global** dynamics of relaxation and self-organization

\[ S.-P. \]
\[ V = V_A / Rm^{1/2} \]

- ⬤ - how describe **global** dynamics of relaxation and self-organization

→ multiple, interacting/overlapping reconnection events

→ turbulence, stochastic lines, etc
What does ‘Self-Organization’ mean?

- context: driven, dissipative, open system
- turbulence/stochasticity - multiple reconnection states
- Profile state (resilient, stiff) attractors
- usually, multiple energy channels possible
- bifurcations between attractor states possible
- attractor states macroscopically stable, though may support microturbulence

Elements of Theory

- universality (or claims thereof)
- coarse graining - i.e., diffusion
- constraint release - i.e., relaxation of freezing-in law
- selective decay hypothesis
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<td>nearly marginal $m = 1$ ’s + resistive interchange +...</td>
<td>ITG, CTEM, ... Issue: ELMs?! (domain limited)</td>
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- **Universality**:

Taylor State (Clear)

\[ H_M = \int d^3 x A \cdot B \]

only constraint

Magnetic energy dissipated as

\[ H_M \text{ conserved} \]

Profile Consistency (especially pedestal) (soft)

PV mixed, subject dynamical constraints

Enstrophy (Turbulence) mixed, dissipated, as macroscopic flow emerges
Why Principles?

→ INSIGHT

→ Physical ideas necessary to guide both physical and digital experiments

→ Principles + Reduced Models required to extract and synthesize lessons from case-by-case analysis

→ Principles guide approach to problem reduction
Examples of Self-Organization Principles

→ Turbulent Pipe Flow: (Prandtl → She)

\[
\sigma = -\nu_T \frac{\partial \langle v_y \rangle}{\partial x} \quad \nu_T \sim v_* x \quad \Rightarrow \langle v_y \rangle \sim v_* \ln x
\]

Streamwise Momentum undergoes scale invariant mixing

→ Magnetic Relaxation: (Woltjer-Taylor)

(RFP, etc) (Focus 1) Minimize \( E_M \) at conserved global \( H_M \) \( \Rightarrow \) Force-Free RFP profiles

→ PV Homogenization/Minimum Enstrophy: (Taylor, Prandtl, Batchelor, Bretherton, ...)

(Focus 2) → PV tends to mix and homogenize

→ Flow structures emergent from selective decay of potential enstrophy relative energy

→ Shakura-Sunyaev Accretion

→ disk accretion enabled by outward viscous angular momentum flux
- Will show many commonalities - though NOT isomorphism - of magnetic and flow self-organization

- Will attempt to expose numerous assumptions in theories thereof

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II.) Focus I - Magnetic Relaxation

Prototype of RFP’s: 

- **Zeta**  
  (UK: late 50’s - early 60’s)

- toroidal pinch = vessel + gas + transformer
- initial results → violent macro-instability, short life time
- weak $B_T$ → stabilized pinch ↔ sausage instability eliminated
- $I_p > I_{p, crit}$ ( $\theta > 1+$ ) → access to “Quiescent Period”

Properties of Quiescent Period:

- macrostability - reduced fluctuations
- $\tau_E \sim 1 \text{ msec} \quad T_e \sim 150\text{eV}$
- $B_T(a) < 0$ → reversal

Quiescent Period is origin of RFP
Further Developments

- Fluctuation studies:
  \[ m = 1 \quad \text{kink-tearing} \rightarrow \text{tend toward force-free state} \]
  \[ \text{turbulence} = \text{resistive interchange, ...} \]

- Force-Free Bessel Function Model
  \[ B_\theta = B_0 J_1(\mu r) \quad B_z = B_0 J_0(\mu r) \]
  \[ J = \alpha B \]
  observed to correlate well with observed B structure

- L. Woltjer (1958) : Force-Free Fields at constant \( \alpha \)
  \[ \rightarrow \text{follows from minimized } E_M \text{ at conserved } \int d^3x A \cdot B \]
  - steady, albeit modest, improvement in RFP performance, operational space
  \[ \rightarrow \text{Needed: Unifying Principle} \]
Theory of Turbulent Relaxation  (J.B. Taylor, 1974)

→ hypothesize that relaxed state minimizes magnetic energy subject to constant 
global magnetic helicity

i.e. profiles follow from: \[
\delta \left[ \int d^3 x \frac{B^2}{8\pi} + \lambda \int d^3 x A \cdot B \right] = 0
\]

\[\nabla \times B = \mu B ; \quad J_{||}/B = \frac{J \cdot B}{B^2} = \text{const}
\]

Taylor state is:

- force free
- flat/homogenized \(J_{||}/B\)
- recovers BFM, with reversal for \(\theta = \frac{2I_p}{aB_0} > 1.2\)

- Works amazingly well
Result:

and numerous other success stories

→ Questions:

- what is magnetic helicity and what does it mean?
- why only global magnetic helicity as constraint?
- Theory predicts end state → what can be said about dynamics?
- What does the pinch say about dynamics?

→ Central Issue: Origin of Irreversibility

\[ \theta = \mu a / 2 = \frac{2 I_p}{a B_0} \]

\[ F = \frac{B_{z,wall}}{\langle B \rangle} \]
**Magnetic helicity - what is it?**

- consider two linked, closed flux tubes

  **Tube 1:** Flux $\phi_1$, contour $C_1$

  **Tube 2:** Flux $\phi_2$, contour $C_2$

if consider tube 1: \[ H_M^1 = \int_{V_1} d^3x \mathbf{A} \cdot \mathbf{B} = \oint_{C_1} dl \int_{A_1} dS \mathbf{A} \cdot \mathbf{B} = \oint_{C_1} dl_1 \cdot \mathbf{A} \int_{A_1} da \cdot \mathbf{B} = \phi_1 \oint_{C_1} dl_1 \cdot \mathbf{A} = \phi_1 \phi_2 \]

similarly for tube 2: \[ H_M^2 = \phi_1 \phi_2 \]

so \[ H_M = 2\phi_1 \phi_2 \]

generally: \[ H_M = \pm 2n\phi_1 \phi_2 \]
- Magnetic helicity measures self-linkage of magnetic configuration

- conserved in ideal MHD - topological invariant

\[
\frac{d}{dt} H_M = -2\eta c \int d^3 x J \cdot B
\]

- consequence of Ohm's Law structure, only

N.B.

- can attribute a finite helicity to each closed flux tube with non-constant \( q(r) \)

- in ideal MHD \( \rightarrow \infty \) number of tubes in pinch. Can assign infinitesimal tube to each field line

- \( \infty \) number of conserved helicity invariants

\( \rightarrow \) Follows from freezing in
Question:

How many magnetic field lines in the universe?

(E. Fermi to M.N. Rosenbluth, oral exam at U. Chicago, late 1940’s...)
Why Global helicity, Only?

- in ideal plasma, helicity conserved for each line, tube

\[ \mathbf{J} = \mu(\alpha, \beta) \mathbf{B} \quad \mu(\alpha', \beta') \neq \mu(\alpha, \beta) \]

- Turbulent mixing eradicates identity of individual flux tubes, lines!

\[ \text{i.e.} \]

- if turbulence s/t field lines stochastic, then ‘1 field line’ fills pinch.

1 line $\leftrightarrow$ 1 tube $\rightarrow$ only global helicity meaningful.

- in turbulent resistive plasma, reconnection occurs on all scales, but: $\tau_R \sim l^\alpha \quad \alpha > 0$

\[ (\alpha = 3/2 \quad \text{for S-P reconnection}) \]

Thus larger tubes persist longer: Global flux tube most robust

- selective decay: absolute equilibrium stat. mech. suggests possibility of inverse cascade of magnetic helicity (Frisch ’75) $\rightarrow$ large scale helicity most rugged.
Comments and Caveats

→ Taylor’s conjecture that global helicity is most rugged invariant remains a conjecture
  
  → unproven in any rigorous sense

→ many attempts to expand/supplement the Taylor conjecture have had little lasting impact (apologies to some present....)

→ Most plausible argument for global $H_M$ is stochastization of field lines → forces confinement penalty. No free lunch!

→ Bottom Line:
  
  - Taylor theory, simple and successful
  
  - but, no dynamical insight!
- The question of Dynamics brings us to mean field theory (c.f. Moffat ’78 and an infinity of others - see D. Hughes, Thursday Lecture)

- Mean Field Theory → how represent $\langle \tilde{v} \times \tilde{B} \rangle$?

  → how relate to relaxation?

- Caveat: - MFT assumes fluctuations are small and quasi-Gaussian. They are often NOT

- MFT is often very useful, but often fails miserably

- Structural Approach (Boozer): (plasma frame)

  $$\langle E \rangle = \eta \langle J \rangle + \langle S \rangle$$

  → something → related to $\langle \tilde{v} \times \tilde{B} \rangle$

  $\langle S \rangle$ conserves $H_M$

  $\langle S \rangle$ dissipates $E_M$

  Note this is ad-hoc, forcing $\langle S \rangle$ to fit the conjecture. Not systematic, in sense of perturbation theory
Now

\[ \partial_t H_M = -2c\eta \int d^3x \langle J \cdot B \rangle - 2c \int d^3x \langle S \cdot B \rangle \]

\[ \therefore \langle S \rangle = \frac{B}{B^2} \nabla \cdot \Gamma_H \]

Conservation \( H_M \) \( \rightarrow \) \( \langle S \rangle \sim \nabla \cdot (\text{Helicity flux}) \)

\[ \partial_t \int d^3x \frac{B^2}{8\pi} = - \int d^3x \left[ \eta J^2 - \Gamma_H \cdot \nabla \frac{\langle J \rangle \cdot B}{B^2} \right] \]

so

\[ \Gamma_H = -\lambda \nabla (J_\parallel / B) \text{, to dissipate } E_M \]

\( \rightarrow \) simplest form consistent with Taylor hypothesis

\( \rightarrow \) turbulent hyper-resistivity \( \lambda = \lambda[\langle \tilde{B}^2 \rangle] \) - can derive from QLT

\( \rightarrow \) Relaxed state: \( \nabla (J_\parallel / B) \rightarrow 0 \) homogenized current \( \rightarrow \) flux vanishes
Dynamics II: The Pinch’s Perspective

- Boozer model not based on fluctuation structure, dynamics

- Aspects of hyper-resistivity do enter, but so do other effects

  → Point: Dominant fluctuations controlling relaxation are m=1 tearing modes resonant in core → global structure

  → Issue: What drives reversal \( B_z \) near boundary?

Approach: QL \( \langle \tilde{v} \times \tilde{B} \rangle \) in MHD exterior - exercise: derive!

\[
\langle \tilde{v} \times \tilde{B} \rangle \approx \sum_k \gamma_k \frac{R}{r} (q_{res} - q(r)) \langle B_\theta \rangle \partial_r (|\xi_r|^2)
\]

i.e. \( \langle J_\theta \rangle \) driven opposite \( \langle B_\theta \rangle \) → drives/sustains reversal
→ What of irreversibility - i.e. how is kink-driven reversal ‘locked-in’?

![Diagram showing magnetic surfaces and r_rev]

→ drive \( J/|B| \) flattening, so higher n’s destabilized by relaxation front

→ global scattering → propagating reconnection front

\[
\begin{align*}
\text{m=1,} & \quad n \\
\text{m=1,} & \quad n+1 \\
\text{m=0,} & \quad n=1
\end{align*}
\]

→ driven current sheet, at \( r_{rev} \)

(sum beat) \[ \{ \begin{array}{l}
m=2, \\
2n+1
\end{array} \]

(difference beat)

but then \( m=1, \quad n+2 \) driven → tearing activity, and relaxation region, broadens

→ Bottom Line: How Pinch ‘Taylors itself’ remains unclear, in detail
Summary of Magnetic Relaxation

concept: topology

process: stochastization of fields, turbulent reconnection

constraint released: local helicity

players: tearing modes

Mean Field: \( \text{EMF} = \langle \vec{v} \times \vec{B} \rangle \)

Global Constraint: \( \int d^3 x A \cdot B \)

NL: Helicity Density Flux

Outcome: B-Profile

Shortcoming: Rates, confinement \( \rightarrow \) turbulent transport
Focus II: Potential Vorticity Mixing ↔ Iso-vorticity Contour Reconnection

→ Prandtl-Batchelor Theorem and PV Homogenization

→ Self-Organization of Zonal Flows
PV and Its Meaning: Representative Systems

The Fundamentals

- Kelvin’s Theorem for rotating system

\[ \omega \to \omega + 2\Omega \]

\[ \begin{array}{c} \text{relative} \\ \text{planetary} \end{array} \]

\[ \int \mathbf{v} \cdot d\mathbf{l} = \int d\mathbf{a} \cdot (\omega + 2\Omega) \equiv C \]

\[ \dot{C} = 0 \text{, to viscosity (vortex reconnection)} \]

- \( Ro = \frac{V}{2\Omega L} \ll 1 \)

\[ \mathbf{V} \cong -\nabla_\perp p \times \hat{z}/(2\Omega) \]

\[ \text{geostrophic balance} \]

\[ \to 2D \text{ dynamics} \]

- Displacement on beta plane

\[ \dot{C} = 0 \]

\[ \frac{d}{dt} \omega \cong -\frac{2\Omega}{A} \sin \theta_0 \frac{dA}{dt} \]

\[ = -2\Omega \frac{d\theta}{dt} = -\beta V_y \]

\[ \omega = \nabla^2 \phi, \quad \beta = 2\Omega \sin \theta_0 / R \]
Fundamentals II

- Q.G. equation \[ \frac{d}{dt}(\omega + \beta y) = 0 \]
  
- Locally Conserved PV \[ q = \omega + \beta y \]
  \[ q = \omega / H + \beta y \]

- Latitudinal displacement → change in relative vorticity

- Linear consequence → **Rossby Wave**

  \[ \omega = -\beta k_x / k^2 \]

  observe: \[ v_{g,y} = 2\beta k_x k_y / (k^2)^2 \]

  → Rossby wave intimately connected to momentum transport

- Latitudinal PV Flux → circulation
- **Obligatory re: 2D Fluid**

- \( \omega \) Fundamental:
  \[
  \partial_t \omega = \nabla \times (V \times \omega)
  \]

  \[
  \frac{d}{dt} \frac{\omega}{\rho} = \frac{\omega}{\rho} \cdot \nabla V \quad \rightarrow \text{Stretching}
  \]

- 2D \( d\omega/dt = 0 \) \rightarrow \( E = \langle v^2 \rangle \) conserved

  \[
  \Omega = \langle \omega^2 \rangle
  \]

  **Inverse energy range** \( E(k) \sim k^{-5/3} \)
  **forward enstrophy range** \( E(k) \sim k^{-3} \)

  \[
  \partial_t \langle \Delta k^2 \rangle_E > 0 \quad \text{with} \quad \dot{E} = \dot{\Omega} = 0
  \]

  \[
  \partial_t \langle \Delta k^2 \rangle_E = -\partial_t \bar{k}_E^2
  \]

  \[
  \left\{ \begin{array}{l}
  \partial_t \bar{k}_E^2 < 0 \\
  \partial_t \bar{k}_\Omega^2 > 0
  \end{array} \right\} \rightarrow \text{large scale accumulation}
  \]

  \[
  \rightarrow \text{flow to small scale dissipation}
  \]
Isn’t this Meeting about Plasma?

2 Simple Models

a.) Hasegawa-Wakatani (collisional drift inst.)

\[ \mathbf{V} = \frac{c}{B} \hat{z} \times \nabla \phi + \mathbf{V}_{pol} \]

\[ \sim (\omega/\Omega) \]

\[ L > \lambda_D \rightarrow \nabla \cdot \mathbf{J} = 0 \rightarrow \nabla_\perp \cdot J_\perp = -\nabla_\parallel J_\parallel \]

\[ J_\perp = n|e|V_\parallel \]

\[ J_\parallel : \eta J_\parallel = -(1/c)\partial_t A_\parallel - \nabla_\parallel \phi + \nabla_\parallel p_e \]

b.) Hasegawa-Mima (DW)

\[ \partial_t n_e = 0 \]

\[ \rightarrow \frac{dn_e}{dt} + \frac{\nabla_\parallel J_\parallel}{-n_0|e|} = 0 \]

MHD: \[ \partial_t A_\parallel \text{ v.s. } \nabla_\parallel \phi \]

DW: \[ \nabla_\parallel p_e \text{ v.s. } \nabla_\parallel \phi \]
So \( H-W \)

\[
\rho_s^2 \frac{d}{dt} \nabla^2 \hat{\phi} = -D_\parallel \nabla_\parallel^2 (\hat{\phi} - \hat{n}/n_0) + \nu \nabla^2 \nabla^2 \hat{\phi}
\]

\[
\frac{d}{dt} n - D_0 \nabla^2 \hat{n} = -D_\parallel \nabla_\parallel^2 (\hat{\phi} - \hat{n}/n_0)
\]

\( D_\parallel k_\parallel^2 / \omega \) is key parameter

n.b. \( PV = n - \rho_s^2 \nabla^2 \phi \) \[ \frac{d}{dt} (PV) = 0 \] \( \rightarrow \) total density

b.) \( D_\parallel k_\parallel^2 / \omega \gg 1 \rightarrow \hat{n}/n_0 \sim e\hat{\phi}/T_e \) \( (m, n \neq 0) \)

\[
\frac{d}{dt} (\phi - \rho_s^2 \nabla^2 \phi) + v_* \partial_y \phi = 0 \quad \rightarrow \text{H-M}
\]

n.b. \( PV = \phi - \rho_s^2 \nabla^2 \phi + \ln n_0(x) \)
An infinity of models follow:

- MHD: ideal ballooning resistive → RBM
- HW + $A_{\parallel}$: drift - Alfven
- HW + curv.: drift - RBM
- HM + curv. + Ti: Fluid ITG
- gyro-fluids

N.B.: Most Key advances appeared in consideration of simplest possible models
Homogenization Theory (Prandtl, Batchelor, Rhines, Young)

\[ \partial_t q + \nabla \phi \times \hat{z} \cdot \nabla q = \nu \nabla^2 q \]

Now: \( t \to \infty \quad \partial_t q \to 0 \)

For \( \nu = 0 \quad q = q(\phi) \)

\[ \rightarrow \quad q = q(\phi) \quad \text{is arbitrary solution} \]

\[ \rightarrow \quad \text{can develop arbitrary fine scale} \quad q = q(\phi) \]

\[ \rightarrow \quad \text{closed stream lines,} \quad \nu = 0 \]

\[ \rightarrow \quad \text{no irreversibility} \]
Now $\nu \neq 0$

→ non-diffusive stretching produces arbitrary fine scale structure

→ for small, but finite $\nu$, instead of fine scale structure, must have:

$$q(\phi) \to const \quad t \to \infty \quad \text{small } \nu \to \text{global behavior}$$

i.e. finite $\nu$ at large $Re \to PV$ homogenization

analogy in MHD? → Flux Expulsion
Prandtl - Batchelor Theorem:

Consider a region of 2D incompressible flow (i.e. vorticity advection) enclosed by closed streamline $C_0$. Then, if diffusive dissipation, i.e. $\partial_t q + \nabla \phi \times \hat{z} \cdot \nabla q = \nu \nabla^2 q$
then vorticity $\rightarrow$ uniform (homogenization), as $t \rightarrow \infty$
within $C_0$

$\rightarrow$ underpins notion of PV mixing $\rightarrow$ basic trend

$\rightarrow$ fundamental to selective decay to minimum enstrophy state in 2D fluids (analogue of Taylor hypothesis)
Proof:

\[ \int_{A_n} \nabla \cdot (v q) = 0 \quad \text{(closed streamlines)} \]

\[ 0 = \int_{A_n} \nabla \cdot (v \nabla q) = v \int_{C_n} dl \hat{n} \cdot \nabla q \]

(form of dissipation relevant!)

For \( q = q(\phi) \)

\[ 0 = v \int_{C_n} dl \hat{n} \cdot \nabla \phi_n \frac{\delta q}{\delta \phi_n} = v \frac{\delta q}{\delta \phi_n} \int_{C_n} dl \hat{n} \cdot \nabla \phi_n \]

\[ \therefore 0 = \nu \frac{\delta q}{\delta \phi_n} \Gamma_n \]

\[ \therefore \frac{\delta q}{\delta \phi_n} = 0 \quad \rightarrow \quad q \text{ homogenized, within } C_0 \]

\[ \rightarrow \quad q' \text{ tends to flatten!} \]
How long to homogenize? ↔ What are the time scales?

Key: Differential Rotation within Eddy

Key: synergism between shear and diffusion

\[
1/\tau_{mix} \sim 1/\tau_c (Re)^{-1/3}
\]

\( \tau_c \equiv \text{circulation time} \)

PV homogenization occurs on hybrid decorrelation rate

but \( \tau_{mix} \ll \tau_D \) for \( Re \gg 1 \) → time to homogenize is finite

Point of the theorem is global impact of small dissipation - akin Taylor
PV Transport and Potential Enstrophy Balance → Zonal Flow

Preamble I

• Zonal Flows Ubiquitous for:
  ~ 2D fluids / plasmas $R_0 < 1$
  Rotation $\tilde{Ω}$, Magnetization $\vec{B}_0$, Stratification
  Ex: MFE devices, giant planets, stars…
Preamble II

• What is a Zonal Flow?
  – $n = 0$ potential mode; $m = 0$ (ZFZF), with possible sideband (GAM)
  – toroidally, poloidally symmetric $ExB$ shear flow

• Why are Z.F.’s important?
  – Zonal flows are secondary (nonlinearly driven):
    • modes of minimal inertia (Hasegawa et. al.; Sagdeev, et. al. ‘78)
    • modes of minimal damping (Rosenbluth, Hinton ‘98)
    • drive zero transport ($n = 0$)
  – natural predators to feed off and retain energy released by gradient-driven microturbulence
Heuristics of Zonal Flows a):
Simplest Possible Example: Zonally Averaged Mid-Latitude Circulation

» classic GFD example: Rossby waves + Zonal flow
  (c.f. Vallis ’07, Held ’01)

» Key Physics:
  
  \[
  \omega_k = -\frac{\beta k_x}{k_1^2} \\
  v_{gy} = 2\beta \frac{k_x k_y}{k_1^2} \\
  \therefore v_{gy} v_{phy} < 0 \\
  \Rightarrow \text{Backward wave!}
  \]

\[
\langle \tilde{v}_y \tilde{v}_x \rangle = \sum_k -k_x k_y |\hat{\phi}_k|^2
\]

\[\Rightarrow \text{Momentum convergence at stirring location}\]
...“the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum into this region.” (I. Held, ’01)

- Outgoing waves ⇒ incoming wave momentum flux

![Diagram showing viscous damping and zonal shear layer formation.]

- Local Flow Direction (northern hemisphere):
  - eastward in source region
  - westward in sink region
  - set by $\beta > 0$
  - Some similarity to spinodal decomposition phenomena
    → both ‘negative diffusion’ phenomena
Key Point: Finite Flow Structure requires *separation* of excitation and dissipation regions.

=> Spatial structure and wave propagation within are central.

→ momentum transport by *waves*
Key Elements:

- Waves $\rightarrow$ propagation transports momentum $\leftrightarrow$ stresses
  $\rightarrow$ modest-weak turbulence

- vorticity transport $\rightarrow$ momentum transport $\rightarrow$ Reynolds force
  $\rightarrow$ the Taylor Identity

- Irreversibility $\rightarrow$ outgoing wave boundary conditions

- symmetry breaking $\rightarrow$ direction, boundary condition
  $\rightarrow$ $\beta$

- Separation of forcing, damping regions
  $\rightarrow$ need damping region broads than source region
  $\rightarrow$ akin intensity profile...

All have obvious MFE counterparts...
2) MFE perspective on Wave Transport in DW Turbulence

- localized source/instability drive intrinsic to drift wave structure
- outgoing wave energy flux $\rightarrow$ incoming wave momentum flux $\rightarrow$ counter flow spin-up!
- zonal flow layers form at excitation regions

- couple to damping $\leftrightarrow$ outgoing wave
  i.e. Pearlstein-Berk eigenfunction

\[
v_{gr} = -2 \rho_s^2 \frac{k_\theta k_r v_*}{(1 + k_r^2 \rho_s^2)^2}
\]

- $v_{gr} \quad x > 0 \Rightarrow v_{gr} > 0$
  $v_* < 0 \Rightarrow k_r k_\theta > 0$

- $\langle v_r v_{0E} \rangle = -\frac{c^2}{B^2} |\Phi_k|^2 k_r k_\theta < 0$

radial structure
Heuristics of Zonal Flows b.) cont’d

• So, if spectral intensity gradient → net shear flow → mean shear formation

\[
S_r = v_{gr} \varepsilon \equiv - \frac{2k_r k_\theta V_i \rho_*^2}{(1 + k_\perp^2 \rho^2)} \varepsilon
\]

\[
\langle \tilde{v}_r \tilde{v}_\theta \rangle \approx \sum_k - k_r k_\theta |\phi_k^-|^2
\]

• Reynolds stress proportional radial wave energy flux \( \vec{S} \), mode propagation physics (Diamond, Kim ‘91)

• Equivalently: \( \partial_t E + \nabla \cdot S + (\omega \text{Im}\omega)E = 0 \) (Wave Energy Theorem)
  
  \(-\varepsilon\): Wave dissipation coupling sets Reynolds force at stationarity

• Interplay of drift wave and ZF drive originates in mode dielectric

• Generic mechanism…
Towards Calculating Something: Revisiting Rayleigh from PV Perspective

- G.I. Taylor’s take on Rayleigh criterion

  - consider effect on (zonal) flow by displacement of PV: \( \delta y \)

\[
\frac{\partial}{\partial t} \langle v_x \rangle = \langle \tilde{v}_y \tilde{q} \rangle
\]

\[
\tilde{q} = (\text{PV of vorticity blob at } y) - (\text{mean PV at } y)
\]

\[
\langle q(y) \rangle = \langle q(y_0) \rangle + (y - y_0) \left. \frac{d\langle q \rangle}{dy} \right|_{y_0}
\]

Small displacement

\[
\therefore \frac{\partial}{\partial t} \langle v_x \rangle = -\langle \tilde{v}_y \delta y \rangle \frac{d\langle q \rangle}{dy} = -\left( \partial_t \frac{\langle \tilde{\varepsilon}^2 \rangle}{2} \frac{d\langle q \rangle}{dy} \right)
\]

Flow driven by PV Flux
So, for instability

\[
\begin{cases}
\frac{\partial}{\partial t} \langle \tilde{\varepsilon}^2 \rangle > 0 \quad ; \text{growing displacement} \\
\frac{\partial}{\partial t} \int_{-a}^a dy \langle v_x \rangle = 0 \quad ; \text{momentum conservation}
\end{cases}
\]

\[
- \int_{-a}^a dy \left( \frac{\partial_t \langle \tilde{\varepsilon}^2 \rangle}{2} \right) \frac{d\langle q \rangle}{dy} = 0 \quad \frac{d\langle q \rangle}{dy} \text{ must change sign within flow interval} \\
\Rightarrow \text{ inflection point}
\]

also,

\[
\frac{\partial}{\partial t} \left\{ \langle v_x \rangle + \frac{\langle \tilde{\varepsilon}^2 \rangle}{2} \frac{d\langle q \rangle}{dy} \right\} = 0 \quad \tilde{q} = -\tilde{\varepsilon} \frac{d\langle q \rangle}{dy}
\]

\[
\frac{\partial}{\partial t} \left\{ \langle v_x \rangle - \left( -\frac{\langle \tilde{q}^2 \rangle}{2 \partial_2 \langle q \rangle / \partial y} \right) \right\} = 0 \quad -\langle \tilde{q}^2 \rangle / 2 \partial_2 \langle q \rangle / \partial y \equiv \\
Pseudomomentum for QG system
\]

\[\rightarrow\] no slip condition of flow + quasi-particle gas

\[\rightarrow\] (significant) step toward momentum theorem

i.e. ties flow to wave momentum density
Zonal Flows I

• Fundamental Idea:
  – Potential vorticity transport + 1 direction of translation symmetry
    → Zonal flow in magnetized plasma / QG fluid
  – Kelvin’s theorem is ultimate foundation

• G.C. ambipolarity breaking → polarization charge flux → Reynolds force
  – Polarization charge
    \[ \rho^2 \nabla^2 \phi = n_{i,GC}(\phi) - n_e(\phi) \]
    (polarization length scale)
  – so \( \Gamma_{i,GC} \neq \Gamma_e \) \[ \rho^2 \left\langle \tilde{v}_{rE} \nabla^2 \tilde{\phi} \right\rangle \neq 0 \] ‘PV transport’
  – polarization flux \( \rightarrow \) What sets cross-phase?
  – If 1 direction of symmetry (or near symmetry):
    \[ -\rho^2 \left\langle \tilde{v}_{rE} \nabla^2 \tilde{\phi} \right\rangle = -\partial_r \left\langle \tilde{v}_{rE} \tilde{v}_{\perp E} \right\rangle \] (Taylor, 1915)
    \[ -\partial_r \left\langle \tilde{v}_{rE} \tilde{v}_{\perp E} \right\rangle \] Reynolds force \( \rightarrow \) Flow
Notable by Absence: Three “Usual Suspects”

- “Inverse Cascade”
  - Wave mechanism is essentially linear
    → scale separation often dubious
  - PV transport is sufficient / fundamental

- “Rhines Mechanism”
  - requires very broad dynamic range
  - Waves ⇔ $k_R$ ⇔ forced strong turbulence
  - strong turbulence model

- “Modulational Instability”
  - coherent, quasi-coherent wave process
  - useful concept, but not fundamental

Lesson: Formation of zonal bands is *generic* to the response of a rapidly rotating fluid to any localized perturbation

→ see P.D. et al. PPCF’05, CUP’10 for detailed discussion
Inverse Cascade/Rhines Mechanism

\[ \omega_k \sim -\beta k_x / k^2 \]

1/\tau_k

transfer \( \Leftrightarrow \) triad couplings

\[ \begin{align*}
  k & \quad k' \\
  k'' & \quad k
\end{align*} \]

eddy transfer: \( \omega_{MM} < 1 / \tau_c \)

wave transfer: \( \omega_{MM} > 1 / \tau_c \)

cross over: \( \omega_{MM} \sim 1 / \tau_c \)

\[ l_R \sim (\tilde{v} / \beta)^{1/2} \sim \epsilon^{1/5} / \beta^{3/5} \]

Rhines Scale - emergent characteristic scale for ZF

Contrast: Rhines mechanism vs critical balance

The crux:

- 3 wave resonance requires 1 wave with \( k_x = 0 \)

- ZF’s appear at \( k_R \)

- coupling maximal at \( k_R \)

\[ k_R \quad \text{Z.F. dominates} \]
→ **Caveat Emptor:**

- often said `'Zonal Flow Formation ≈ Inverse Cascade’`

  but

- anisotropy crucial → \( \langle \tilde{V}^2 \rangle, \beta, \) forcing → ZF scale

- numerous instances with:  
  - no inverse inertial range
  - ZF formation ↔ quasi-coherent

all really needed:

\[
\langle \tilde{V}_y \tilde{q} \rangle \rightarrow \text{PV Flux} \rightarrow \langle \tilde{V}_y \tilde{V}_x \rangle \rightarrow \text{Flow}
\]

→ transport and mixing of PV are fundamental elements of dynamics
Zonal Flows II

- Potential vorticity transport and momentum balance
  - Example: Simplest interesting system → Hasegawa-Wakatani
    - Vorticity: $\frac{d}{dt} \nabla^2 \phi = -D_\parallel \nabla^2 (\phi - n) + D_0 \nabla^2 \nabla^2 \phi$
    - Density: $\frac{dn}{dt} = -D_\parallel \nabla^2 (\phi - n) + D_0 \nabla^2 n$
  - Locally advected PV: $q = n - \nabla \phi^2$
    - PV: charge density
      - $n \rightarrow$ guiding centers
      - $-\nabla \phi^2 \rightarrow$ polarization
    - conserved on trajectories in inviscid theory $\frac{dq}{dt} = 0$
    - PV conservation $\rightarrow$ Freezing-in law
      - Kelvin’s theorem $\rightarrow$ Dynamical constraint

D_0 classical, feeble
Pr = 1 for simplicity
Zonal Flow II, cont’d

- Potential Enstrophy (P.E.) balance
  \[
  \frac{d}{dt} \langle q^2 \rangle = 0 \quad \text{flux dissipation} \quad \langle \rangle \rightarrow \text{coarse graining}
  \]
  \[
  \text{LHS} \Rightarrow \frac{d}{dt} \langle \tilde{q}^2 \rangle = \partial_t \langle \tilde{q}^2 \rangle + \partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle + D_0 \langle (\nabla \tilde{q})^2 \rangle
  \]
  \[
  \text{RHS} \Rightarrow \text{P.E. evolution} - \langle \tilde{V}_r \tilde{q} \rangle \langle q \rangle' \Rightarrow \text{P.E. Production by PV mixing / flux}
  \]

- PV flux:
  \[
  \langle \tilde{V}_r \tilde{q} \rangle = \langle \tilde{V}_r \tilde{n} \rangle - \langle \tilde{V}_r \nabla^2 \phi \rangle; \quad \text{but} \quad \langle \tilde{V}_r \nabla^2 \phi \rangle = \partial_r \langle \tilde{V}_r \tilde{V}_\theta \rangle
  \]
  \[
  \vdash \text{P.E. production directly couples driving transport and flow drive}
  \]

- Fundamental Stationarity Relation for Vorticity flux
  \[
  \langle \tilde{V}_r \nabla^2 \phi \rangle = \langle \tilde{V}_r \tilde{n} \rangle + (\delta_t \langle \tilde{q}^2 \rangle) / \langle q \rangle'
  \]
  \[
  \text{Reynolds force} \quad \text{Relaxation} \quad \text{Local PE decrement}
  \]
  \[
  \vdash \text{Reynolds force locked to driving flux and P.E. decrement; transcends quasilinear theory}
  \]
Contrast: Implications of PV Freezing-in Law

\[
\begin{align*}
\frac{dn}{dt} &= 0 \quad (?) \\
\frac{d\langle n \rangle}{dr} &\neq 0 \\
\tilde{n} \text{ grows} &\rightarrow \langle \tilde{V}_r \tilde{n} \rangle \rightarrow :-( 
\end{align*}
\]

\[
\begin{align*}
\frac{dq}{dt} &= 0 \quad (!) \\
\frac{d\langle q \rangle}{dr} &\neq 0 \\
\tilde{q} \text{ grows} &\rightarrow \left\{ \begin{array}{l}
\langle \tilde{V}_r \tilde{n} \rangle \rightarrow \text{transport} \rightarrow :-( \\
\langle \tilde{V}_r \nabla^2 \phi \rangle \rightarrow \text{flow} \rightarrow :-) 
\end{array} \right.
\end{align*}
\]

Lesson: Even if \( \langle q \rangle \cong \langle n \rangle \), PV conservation must channel free energy into zonal flows!

Key Question: Branching ratio of energy coupled to flow vs transport-inducing fluctuations?
Combine: \[
\begin{align*}
\text{PE balance} & \quad \partial_t \langle V_\theta \rangle = -\langle \tilde{V}_r \nabla^2 \phi \rangle - \nu \langle V_\theta \rangle \\
& \quad \text{yields...}
\end{align*}
\]

Charney-Drazin Momentum Theorem

(1960, et.seq., P.D., et.al. ’08, for HW)

\[
\begin{align*}
\text{Pseudomomentum} & \quad \text{local P.E. decrement} \\
\Rightarrow \quad \partial_t \{ \langle WAD \rangle + \langle V_\theta \rangle \} = & \quad -\langle \tilde{V}_r \tilde{n} \rangle - \delta_t \langle \tilde{q}^2 \rangle / \langle q \rangle' - \nu \langle V_\theta \rangle \\
& \quad \text{driving flux} \quad \text{drag}
\end{align*}
\]

WAD = Wave Activity Density, \( \langle \tilde{q}^2 \rangle / \langle q \rangle' \)

- pseudomomentum in \( \theta \)-direction (Andrews, McIntyre ’78)
- Generalized Wave Momentum Density

\[
\begin{align*}
i) \quad \text{momentum of quasi-particle gas of waves, turbulence} \\
ii) \quad \text{consequence of azimuthal/poloidal symmetry} \\
iii) \quad \text{not restricted to linear response, but reduces correctly}
\end{align*}
\]
What Does it Mean? → “Non-Acceleration Theorem”:

\[
\partial_t \{ (WAD) + \langle V_\theta \rangle \} = -\langle \tilde{V}_r \tilde{n} \rangle - \delta_t \langle \tilde{q}^2 \rangle / \langle q \rangle' - \nu \langle V_\theta \rangle
\]

- absent \{\n  \langle \tilde{V}_r \tilde{n} \rangle, \text{ driving flux} \\
  \delta_t \langle \tilde{q}^2 \rangle, \text{ local potential enstrophy decrement}
\}\n
→ cannot \{\n  accelerate \\
  maintain
\}\n
Z.F. with stationary fluctuations!

Essential physics is PV conservation and translational invariance in \( \theta \) → freezing quasi-particle gas momentum into flow → relative “slippage” required for zonal flow growth

obvious constraint on models of stationary zonal flows!

↔ need explicit connection to relaxation, dissipation

N.B. Inhomogeneous dissipation → incomplete homogenization!?
Aside: H-M

\[ \partial_t \{ \text{WAD} + \langle V_\theta \rangle \} = \frac{\langle \tilde{f}^2 \rangle \tau_c}{\langle q' \rangle'} - \frac{1}{\langle q' \rangle'} \left\{ \partial_r \langle \tilde{V}_r \delta q^2 \rangle + \mu \langle (\nabla \delta q)^2 \rangle \right\} - \nu \langle V_\theta \rangle \]

C-D prediction for \( \langle V_\theta \rangle \) at stationary state, HM model

\[ \langle V_\theta \rangle = \frac{1}{\nu \langle q' \rangle'} \left\{ \langle \tilde{f}^2 \rangle \tau_c - \partial_r \langle \tilde{V}_r \delta q^2 \rangle + \mu \langle (\nabla \delta q)^2 \rangle \right\} \]

→ Note: Flow direction set by: \( \langle q' \rangle' \), source, sink distribution
→ Forcing, damping profiles determine shear
→ Potential Enstrophy Transport impact flow structure
In More Depth: What Really Determines Zonal Flow?

- driving flux: \( \langle \tilde{V}_r \tilde{n} \rangle = \Gamma_0 - \Gamma_{\text{col}} = \int dr' S_n(r') - \Gamma_{\text{col}} \)
- Total flux \( \Gamma_0 \) fixed by sources, \( S_n \rightarrow \) flux driven system
- Collisional flux in turbulent system, \( \Gamma_{\text{col}} \) (computed with actual profiles)
- \( \Gamma_0 - \Gamma_{\text{col}} \rightarrow \) available flux

- P.E. decrement: \( \delta_t \langle \tilde{q}^2 \rangle = \partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle + D_0 \langle (\nabla \tilde{q})^2 \rangle \)
  \( \rightarrow \) change in roton intensity (PE) changes flow profile
  - roton dissipation
  - P.E. flux, direction increment, according to convergence (> 0) or divergence (< 0) of pseudomomentum, locally

So: P.E. transport and “spreading” intrinsically linked to flow structure, dynamics

Net \( \delta(\text{P.E.}) \) can generate net spin-up

\[ \therefore \] Zonal flow dynamics intrinsically “non-local” \( \leftrightarrow \) couple to turbulence spreading (fast, meso-scale process)
Clarifying the Enigma of Collisionless Zonal Flow Saturation

- Flow evolution with: \( \nu \to 0, S_n \neq 0 \) and nearly stationary turbulence

\[
\partial_t \langle V_\theta \rangle = - \left( \int dr' S_n(r') - \Gamma_{\text{col}} \right) - \left( \partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle + D_0 \langle (\nabla \tilde{q})^2 \rangle / \langle q \rangle' \right)
\]

Possible Outcomes:

- \( \langle q \rangle' \to 0 \), locally \( \to \) shear flow instability (the usual)
  \( \leftrightarrow \) limit cycle of burst and recovery, effective viscosity?
  \( \to \) problematic with magnetic shear
- \( \langle \tilde{V}_r \tilde{n} \rangle \) v.s. \( \partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle \to \) potential enstrophy transport and inhomogeneous turbulence, with \( \tilde{n}/n \sim M.L.T \)
  \( \to \) flux drive vs. roton population flux
  \( \to \) novel saturation mechanism
- \( \langle q \rangle' \to 0 \), globally \( \to \) homogenized PV state (Rhines, Young, Prandtl, Batchelor)
  \( \to \) decouples mean PV, PE evolution
- homogeneous marginality, i.e. \( \int dr' S_n(r') = \Gamma_{\text{col}} \leftrightarrow \) ala' stiff core

N.B.: \( \langle q \rangle' = 0 \Rightarrow \partial_r \langle \tilde{n} \rangle = \partial_r^2 \langle V_E \rangle = \partial_r \langle \omega_E \rangle \to \) particular profile relation!
Summary of Flow Organization

concept: symmetry

process: PV mixing, transport

constraint released: Enstrophy conservation

players: drift waves

Mean Field: \( \Gamma_{PV} = \langle \tilde{v}_r \tilde{q} \rangle \)

Global Constraint: Bounding circulation

NL: Pseudomomentum Flux

Outcome: Zonal Flow Formation

Shortcoming: ZF pattern structure and collisionless saturation
**Summary of comparison**

- Many commonalities between magnetic and flow relaxation apparent.
- Common weak point is limitation of mean field theory → difficult to grapple with strong NL, non-Gaussian fluctuations.

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</table>
Heuristics of Zonal Flows c.)

• One More Way:
• Consider:
  – Radially propagating wave packet
  – Adiabatic shearing field

\[ \frac{d}{dt} k_r = -\frac{\partial}{\partial r} \left( \omega + k_\theta \langle V_{E,ZF} \rangle \right) \Rightarrow \langle k_r^2 \rangle \uparrow \]

\[ \omega_k = \frac{\omega_*}{1 + k^2 \rho_s^2} \downarrow \]

• Wave action density \( N_k = \frac{E(k)}{\omega_k} \) adiabatic invariant
• \( \therefore \ E(k) \downarrow \Rightarrow \text{flow energy decreases, due Reynolds work} \Rightarrow \text{flows amplified (cf. energy conservation)} \)
• \( \Rightarrow \text{Further evidence for universality of zonal flow formation} \)
Heuristics of Zonal Flows d.) cont’d

• Implications:
  – ZF’s generic to drift wave turbulence in any configuration: electrons tied to flux surfaces, ions not
    • g.c. flux → polarization flux
    • zonal flow
  – Critical parameters
    • ZF screening (Rosenbluth, Hinton ‘98)
    • polarization length
    • cross phase → PV mixing

• Observe:
  – can enhance \( e\varphi_{ZF}/T \) at fixed Reynolds drive by reducing shielding, \( \rho^2 \)
  – typically:
    \[
    \frac{\varepsilon}{\varepsilon_0} \approx 1 + \frac{\rho_i^2}{\lambda_D^2} + f_i \frac{\rho_{b,i}^2}{\lambda_D^2} + f_d \frac{\delta_{d,i}^2}{\lambda_D^2}
    \]
    • total screening response
    • banana width
    • banana tip excursion
  – Leverage (Watanabe, Sugama) → flexibility of stellerator configuration
    • Multiple populations of trapped particles
    • \( \langle E_r \rangle \) dependence (FEC 2010)
• Yet more: 
  \[ \frac{\partial}{\partial t} \langle v_\perp \rangle = -\partial_r \langle \tilde{v}_r \tilde{v}_\perp \rangle - \gamma_d \langle v_\perp \rangle + \mu \nabla^2_r \langle v_\perp \rangle \]

• Reynolds force opposed by flow damping

• Damping:
  - Tokamak \( \gamma_d \sim \gamma_{ii} \)
    • trapped, untrapped friction
    • no Landau damping of (0, 0)
  - Stellerator/3D \( \gamma_d \leftrightarrow NTV \)
    • damping tied to non-ambipolarity, also
    • largely unexplored

• Weak collisionality → nonlinear damping – problematic
  → tertiary → ‘KH’ of zonal flow → magnetic shear!?
  → other mechanisms?
4) GAMs Happen

• Zonal flows come in 2 flavors/frequencies:
  – $\omega = 0 \Rightarrow$ flow shear layer
  – GAM $\omega^2 \equiv 2c_s^2 / R^2 (1 + k_r^2 \rho_0^2) \Rightarrow$ frequency drops toward edge $\Rightarrow$ stronger shear
    • radial acoustic oscillation
    • couples flow shear layer (0,0) to (1,0) pressure perturbation
    • $R \equiv$ geodesic curvature (configuration)
    • Propagates radially

• GAMs damped by Landau resonance and collisions
  $\gamma_d \sim \exp[-\omega_{GAM}^2 / (\nu_{thi} / Rq)^2]$
  – $q$ dependence!
  – edge

• Caveat Emptor: GAMs easier to detect $\Rightarrow$ looking under lamp post ?!
Progress I: ZF’s with RMP (with M. Leconte)

- ITER ‘crisis du jour’: ELM Mitigation and Control
- Popular approach: RMP
- ? Impact on Confinement?

Y. Xu ‘11

=> RMP causes drop in fluctuation LRC, suggesting reduced Z.F. shearing
=> What is “cost-benefit ratio” of RMP?

- Physics:
  - in simple H-W model, polarization charge in zonal annulus evolves according:
    \[
    \frac{dQ}{dt} = -\int dA \left[ \tilde{v}_x \tilde{\rho}_{pol} + \left( \frac{\delta B_r}{B_0} \right)^2 D_\parallel \frac{\partial}{\partial x} \left( \langle \phi \rangle - \langle n \rangle \right) \right]_n
    \]
  - Key point: \( \delta B_r \) of RMP induces radial electron current \( \rightarrow \) enters charge balance
Progress I, cont’d

- Implications
  - $\delta B_r$ linearly couples zonal $\hat{\phi}$ and zonal $\hat{n}$
  - Weak RMP $\rightarrow$ correction, strong RMP $\rightarrow$ $\langle E_r \rangle_{ZF} \cong -T_e \partial_r \langle n \rangle / |e|$

- Equations:
  $$\frac{d}{dt} \delta n_q + D_T q^2 \delta n_q + ib_q (\delta \phi_q - (1 - c) \delta n_q) - D_{RMP} q^2 (\delta \phi_q - \delta n_q) = 0$$
  $$\frac{d}{dt} \delta \phi_q + \mu \delta \phi_q - a_q (\delta \phi_q - (1 - c) \delta n_q) + \frac{D_{RMP}}{\rho_s^2} (\delta \phi_q - \delta n_q) = 0$$

- Results:
  - $\gamma > \gamma_c (\mu_{\delta B})$
  - $\mu_{\delta B} > 0$
  - $E_{ZF}/\mathcal{E}_L$ vs $\mathcal{E}/\mathcal{E}_L$ for various RMP coupling strengths

Transitions in presence of RMP
Progress II: β-plane MHD (with S.M. Tobias, D.W. Hughes)

Model

• Thin layer of shallow magneto fluid, i.e. solar tachocline
• β-plane MHD ~ 2D MHD + β-offset  i.e. solar tachocline

\[
\partial_t \nabla^2 \phi + \nabla \phi \times \hat{z} \cdot \nabla \nabla^2 \phi - \nu \nabla^2 \nabla^2 \phi = \beta \partial_x \phi + B_0 \partial_x \nabla^2 A + \nabla A \times \hat{z} \cdot \nabla \nabla^2 A + \tilde{f}
\]

\[
\partial_t A + \nabla \phi \times \hat{z} \cdot \nabla A = B_0 \partial_x \phi + \eta \nabla^2 A \hspace{1cm} \tilde{B}_0 = B_0 \hat{x}
\]

• Linear waves: Rossby – Alfven  \( \omega^2 + \omega \beta \frac{k_x}{k^2} - k_x^2 V^2_A = 0 \)  (R. Hide)
Progress II, cont’d

Observation re: What happens?

- Turbulence $\rightarrow$ stretch field $\rightarrow$ $\langle \tilde{B}^2 \rangle \gg B_0^2$ i.e. $\langle \tilde{B}^2 \rangle / B_0^2 \sim R_m$ (ala Zeldovich)

- Cascades: forward or inverse?
  
  - MHD or Rossby dynamics dominant !?

- PV transport: $\frac{dQ}{dt} = -\int dA \langle \tilde{v} \tilde{q} \rangle \rightarrow$ net change in charge content due PV/polarization charge flux

Now $\frac{dQ}{dt} = -\int dA \left[ \langle \tilde{v} \tilde{q} \rangle - \langle \tilde{B}_r \tilde{J}_\parallel \rangle \right] = -\int dA \partial_x \left\{ \langle \tilde{v}_x \tilde{v}_y \rangle - \langle \tilde{B}_x \tilde{B}_y \rangle \right\}$

- Reynolds mis-match
- vanishes for Alfvenized state

PV flux current along tilted lines

Taylor: $\langle \tilde{B}_x \tilde{J}_\parallel \rangle = -\partial_x \langle \tilde{B}_x \tilde{B}_y \rangle$
Progress II, cont’d

- With Field

\[ B_0 = 10^{-1} \]

\[ B_0 = 10^{-2} \]

\[ B_0 = 0 \]

\[ B_0 = 10^{-3} \]
Progress II, cont’d

- Control Parameters for $\tilde{B}$ enter Z.F. dynamics
  Like RMP, Ohm’s law regulates Z.F.

- Recall
  - $\langle \tilde{v}^2 \rangle$ vs $\langle \tilde{B}^2 \rangle$
  - $\langle \tilde{B}^2 \rangle \sim B_0^2 R_m$ $\rightarrow$ origin of $B_0^2 / \eta$ scaling !?

- Further study $\rightarrow$ differentiate between:
  - cross phase in $\langle \tilde{v}, \tilde{q} \rangle$ and O.R. vs J.C.M
  - orientation: $\vec{B} \parallel \vec{V}$ vs $\vec{B} \perp \vec{V}$
  - spectral evolution

+ = zonal flow state
$\Diamond$ = no zonal flow state
ZF observed

No ZF observed

$\eta$