**Definition**

Let $M = (E, \mathcal{I})$ be a matroid.

- A basis is a maximal independent set.
- A circuit is a minimal dependent set.

- The rank of $A \subseteq E$ is $\max \{|X| : X \subseteq A, X \in \mathcal{I} \}$.
- The rank of $M$ is the rank of $E$.

Recall that all bases have the same rank (by the third property of matroids).

**Example**

Consider the graph $d \xrightarrow{b} e \xrightarrow{a}$

The bases are the spanning trees: $abc, abd, abe, acd, ace$.

The circuits are the minimal cycles: $bcd, bce, de$.

As seen in the last lecture, this matroid is isomorphic to the linear matroid $\{a=(1,0,0), b=(0,1,0), c=(0,0,1), d=(0,1,1), e=(0,1,1)\}$

One can build a central hyper-plane arrangement with $a, b, c, d, e$ as normal vectors.

The arrangement with $H$ as intersection poset.
Definition

Fix a total order $\leq$ on the ground set $E$.

A broken circuit is a subset of $E$ of the form

$$C = C \setminus \{\text{min}_C C\},$$

where $C$ is a circuit.

A broken circuit is an independent set.

Example

$$G = \{\text{d(e,c,a)}\}, \text{ and fix a,b,c,d,e}.\]

<table>
<thead>
<tr>
<th>Circuits</th>
<th>broken circuits</th>
</tr>
</thead>
<tbody>
<tr>
<td>bed</td>
<td>cd</td>
</tr>
<tr>
<td>bce</td>
<td>ce</td>
</tr>
<tr>
<td>de</td>
<td>e</td>
</tr>
</tbody>
</table>

Recall that a simplicial complex $\Delta$ is a set of simplices (line segments, triangles, tetrahedra, ...) such that

- Every face of a simplex from $\Delta$ is also in $\Delta$
- The non-empty intersection of any two simplices of $\Delta$ is a face of $\Delta$ that belong to both simplices

The broken circuit complex is the simplicial complex

$$BC(M) = \{ A \subseteq E : A \text{ contains no broken circuit}\}.$$

Example

$$d(e,c,b)$$

$$BC(M) = \{ A \subseteq E : A \text{ avoids e and cd}\}.$$

Maximal simplices: abc, abd,
The independence complex is the simplicial complex of independent sets. Its simplices of highest dimension are the bases.

Example

\[
\begin{array}{c|c|c}
\text{Dimension} & \# \text{faces} & \text{list} \\
2 & 5 & abc, abd, acd, abe, ace \\
1 & 9 & ab, ac, ad, ae, bc, bd, be, cd, ce \\
0 & 5 & a, b, c, d, e \\
-1 & 1 & \emptyset \\
\end{array}
\]

extended \( f \)-vector: \((15, 9, 5)\)

The characteristic polynomial of a matroid \( M = (E, \mathcal{I}) \) is given by

\[
\chi_M(q) = \sum_{A \subseteq E} (-1)^{|A|} q^{\text{rank}(M) - \text{rank}(A)}
\]
Example

Subsets of $E$ sorted by rank

rank 0 rank 1 rank 2 rank 3

$\emptyset$ $\{a\}$ $\{ab, ac, ad, ae\}$ $\{abc, abd, abe,\}
$\{b\}$ $\{bc, bd, be, cd, ce\}$ $\{acd, ade, abcd,\}$$\{c\}$ $\{ade, bde, cde,\}$ $\{abde, acde,\}$$\{d\}$ $\{bcd, bce, bde,\}$ $\{abce, abde,\}$

Max elements in each rank (called "flits")

$\chi_M(q) = q^3 + (-5 + 1)q^2 + (10 - 5)q + (-5 + 4 - 1)$

$= 1 \cdot q^3 - 4 \cdot q^2 + 5q - 2$

$= \sum_{i \geq 0} (-1)^i f_{i+1}(BC(M)) \cdot q^{\text{rank}(M)-i}$

Theorem

1. For any matroid $M$ and any ordering of the ground set, $\chi_M(q) = \sum_{i \geq 0} (-1)^i f_{i+1}(BC(M)) \cdot q^{\text{rank}(M)-i}$

2. For any graphical matroid that stems out of $G$, $\chi_M$ is the chromatic polynomial of $G$, if $G$ is loopless.

3. For any linear matroid, consider the hyperplane arrangement $A$ of the hyperplanes normal to the vectors of the matroid. Then, $\chi_M = \chi_A$. 

4
Definition

Let $\Delta$ be a simplicial complex. Its $h$-vector is a "compact" version of the $f$-vector, given by

$$
\sum_{k=0}^d f_{k+1} (q-1)^{d-k} = \sum_{k=0}^d h_{k-1} q^{d-k}.
$$

Examples

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$BC(M)$</th>
<th>independence</th>
<th>$M = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(\Delta)$</td>
<td>$(1,4,5,2)$</td>
<td>$(1,5,9,5)$</td>
<td></td>
</tr>
<tr>
<td>$h(\Delta)$</td>
<td>$(1,1,0,0)$</td>
<td>$(1,2,2,0)$</td>
<td></td>
</tr>
</tbody>
</table>

The Tutte polynomial

Many invariants are evaluations of a single invariant:

Definition

The Tutte polynomial of a matroid $M = (E, \mathcal{I})$ is

$$
T_M(x,y) = \sum_{A \subseteq E} (x-1)^{\text{rank}(M) - \text{rank}(A)} (y-1)^{1 - \text{rank}(M)}.
$$
Theorem

For any matroid M, let $r = \text{rank}(M)$. Then,

1. the chromatic polynomial of M is
   \[ \chi_M(q) = (-1)^r T_M(1-r, 0) \]  
   \[ (1) \]

2. the broken circuit complex satisfies
   \[ \sum_{i \geq 0} f_{i-1}(\Delta) q^{r-i} = T_M(q+1, 0) \]  
   \[ (2) \]
   and
   \[ \sum_{i \geq 0} h_{i-1}(\Delta) q^{r-i} = T_M(q, 0) \]  
   \[ (3) \]

3. the independence complex satisfies
   \[ \sum_{i \geq 0} f_{i-1}(\Delta) q^{r-i} = T_M(q+1, 1) \]  
   \[ (4) \]
   and
   \[ \sum_{i \geq 0} h_{i-1}(\Delta) q^{r-i} = T_M(q, 1) \]  
   \[ (5) \]

Proof

(1) $(-1)^r T_M(1-r, 0) = (-1)^r \sum_{A \subseteq E} (-q)^{r - \text{rank}(A)} (-1)^{|A| - \text{rank}(A)}$

\[ = \sum_{A \subseteq E} (-1)^{|A|} q^{r - \text{rank}(4)} \]

\[ = \chi_M(q) \quad \text{(by definition)} \]

(2) Recall that \( \chi_M(q) = \sum_{i \geq 0} f_{i-1}(\text{BC}(M)) q^{r-i} \)

Hence, \[ \sum_{i \geq 0} f_{i-1}(\text{BC}(M)) q^{r-i} = \chi_M(q) \]

\[ = (-1)^r T_M(1-r, 0) \quad \text{by (1)} \]

\[ = (-1)^r \sum_{A \subseteq E} (-q)^{r - \text{rank}(A)} (-1)^{|A| - \text{rank}(A)} \]
\[
= \sum_{A \subseteq E} (-1)^{|A|} q^{r-\text{rank}(A)}
\]

\[
= \sum_{i \geq 0} \left( \sum_{A \subseteq E \atop \text{rank}(A) = i} (-1)^{|A|} q^{r-i} \right)
\]

\[
= \sum_{i \geq 0} (-1)^i \left( \sum_{A \subseteq E \atop \text{rank}(A) = i} (-1)^{|A|-i} q^{r-i} \right)
\]

\[
= \sum_{i \geq 0} f_{i-1}(BC(M)) q^{r-i}
\]

On the other hand

\[
T_m(1,q,0) = \sum_{A \subseteq E} (-1)^{|A|} q^{r-\text{rank}(A)}
\]

\[
= \sum_{i \geq 0} \sum_{A \subseteq E \atop \text{rank}(A) = i} (-1)^{|A|-i} q^{r-i}
\]

\[
= \sum_{i \geq 0} f_{i-1}(BC(M)) q^{r-i}
\]

(3) From (2), we have

\[
T_m(1,q,0) = \sum_{i \geq 0} f_{i-1}(BC(M)) q^{r-i}
\]

Set \( q = q^{-1} \)

\[ 2 \sum_{i \geq 0} f_{i-1}(BC(M)) q^{r-i} \]

\[ = \sum_{i \geq 0} f_{i-1}(BC(M))(q'-1)^{r-i} \]

\[ = \sum_{i \geq 0} h_{i-1}(BC(M)) q^{r-i} \]

\[ = \sum_{i \geq 0} h_{i-1}(BC(M))(q+1)^{r-i} \]

Therefore, \( T_m(q,0) = \sum_{i \geq 0} h_{i-1}(BC(M)) q^{r-i} \)
(4) \( T_M(q+1, 1) = \sum_{A \in E} q \cdot \text{r-rank}(A) \cdot \text{IA1-rank}(A) \)

\[
= \sum_{A \in E} q \cdot \text{r-rank}(A) \\
\begin{cases}
1 & \text{if } \text{IA1-rank}(A) (i.e. A \notin \mathcal{E}) \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \sum_{i \geq 0} f_{i-1}(\Delta) q^{i}, \quad \text{where } \Delta \text{ is the independence complex.}
\]

(5) Proceed exactly like (3), using (4).

Example: \( M = \Delta \)

\[
T_M(x, y) = (x-1)^3 + (x+1)(x-1)^2 + (q+5)(y-1)(y-1)^2 (x-1) + (q+5)(y-1)(y+1)^2)
\]

\[
= x^3 + x^2y + xy^2 + x + y
\]

Consequences:

\[
X_M(q) = (-1)^3 \cdot T_M(1-q, 0) = (-1)^3 (1-q)^3 + (1-q)^2
\]

\[
= q^3 + 4q^2 - 5q - 2
\]

**f-vector of BC(M):**

\[
T_M(q+1, 0) = (q+1)^3 + (q+1)^2
\]

\[
= q^3 + 4q^2 + 5q + 2
\]

\[
= \sum_{i \geq 0} f_{i-1}(B(M)) q^{3-i}
\]

\[
\Rightarrow f-vector(BC(M)) = (1, 4, 5, 2)
\]

**h-vector of BC(M):**

\[
T_M(q, 0) = q^3 + q^2 = \sum_{i \geq 0} h_{i-1}(BC(M)) q^{3-i}
\]

\[
\Rightarrow h-vector(BC(M)) = (1, 1, 0, 0)
\]

**f-vector of I:**

\[
T_M(q+1, 1) = (q+1)^3 + 2(q+1)^2 + 2(q+1)
\]

\[
= q^3 + 5q^2 + 9q + 5
\]

\[
\Rightarrow f-vector of I : (1, 5, 9, 5) \]
h-vector of \( I : \text{TM}(9,1) = q^3 + 2q^2 + 2q \)

\[ \Rightarrow \text{h-vector of } I : (1,2,2,0) \]

Five sequences of numbers from matroids.

Consider an integer sequence \( a = (a_1, a_2, \ldots, a_n) \). It is said to be

- **unimodal** if there exists \( k \in \{1, \ldots, n\} \) such that
  \[ a_1 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots \geq a_n \]

- **log-concave** if, for all \( k \in \{2, \ldots, n-1\} \),
  \[ a_k^2 \geq a_{k-1} a_{k+1} \]

- **flawless** if \( a_i; a_j \Rightarrow a_k \Rightarrow a_0 \) for all \( 1 \leq i \leq k \leq j \leq n \).

- **top-heavy** if \( a_i \leq a_{d-i} \) for all \( 0 \leq i \leq \frac{d}{2} \), where \( d \) is the largest index such that \( a_d \neq 0 \).

Note that log-concave \( \Rightarrow \) unimodal \( \Leftrightarrow \) flawless.

Conjectures (several people, between 1968 and 2003)

The following sequences are unimodal, log-concave and top heavy:

a) The \( f \)-vector of the independence complex

b) The \( h \)-vector of the independence complex

c) The \( f \)-vector of the broken circuit complex

d) The \( h \)-vector of the broken circuit complex

e) The number of maximal subsets of each rank
   (equivalently, the intersections at each rank of the intersection poset if the arrangement is linear, or the number of flats).
Theorems (2012-2022; Huh, in different papers, with several collaborators; Adiprasito, Ardila, Brodén, Denham, Katz, Mathur, Proudfoot, Wang)

- The sequences $a_5, b_5, c_5, d_5$ are unimodal, log-concave, top-heavy and flawless.
- The sequence $e_5$ is top-heavy and flawless.

Conjectures (open).

[Rota, 1971] Sequence $e_1$ is unimodal.

[Mason, 1972] Sequence $e_1$ is log-concave.

References: [AM23], [Eur23]