Math 108 - Geometric Combinatorics

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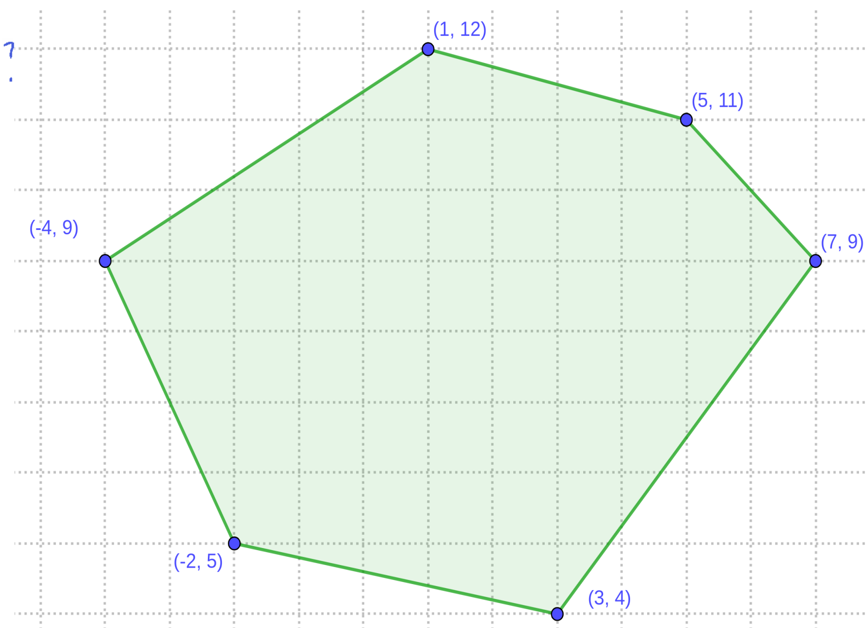
Volumes of polytopes - Pick's theorem

1/25/2023

Given a polytope, how can we compute its volume?



Dimension 1

A polytope is a line segment: $[a,b]$, and its "volume" (i.e. $\underbrace{\text{length}}_{\text{volume in dimension 1}}$) is $b-a$

Dimension 2

The "volume" is the area.

We know how to compute areas of polygons, but the process can be tedious for polygons with a lot of sides

Example: hexagon above.

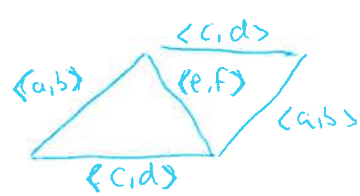
Observation 1: One can always decompose a polygon into triangles, without adding vertices. This process is called "triangulation".

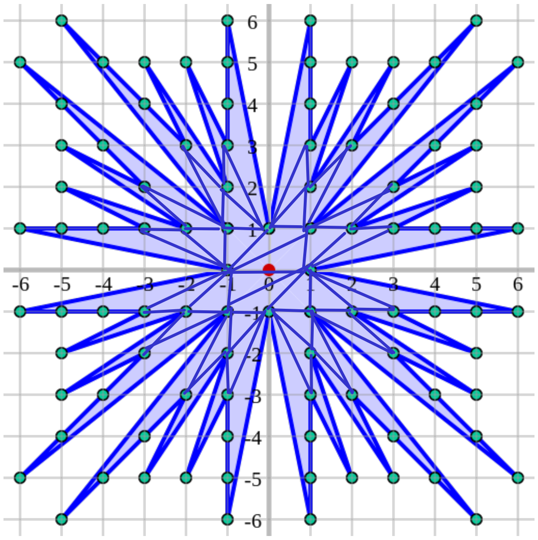
Observation 2: The area of the triangle bounded by the vectors $\langle a,b\rangle, \langle c,d\rangle, \langle e,f\rangle$ is

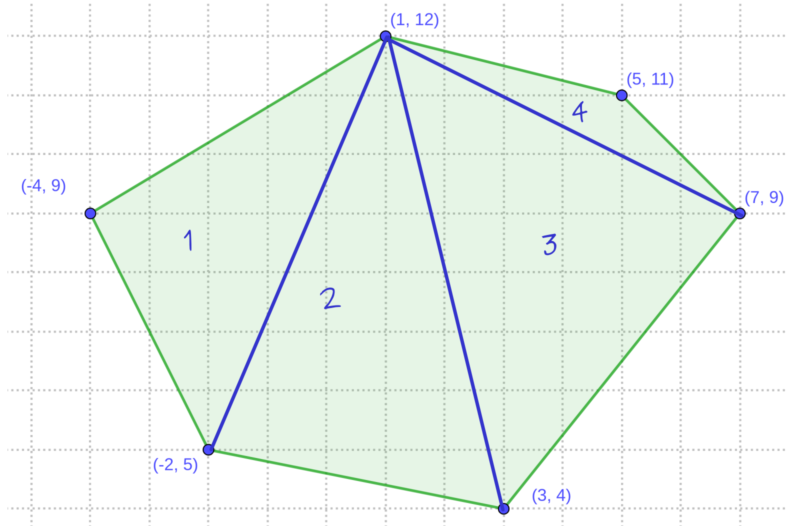
$$\frac{1}{2}\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$$

Area of parallellipiped.

$$\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$$

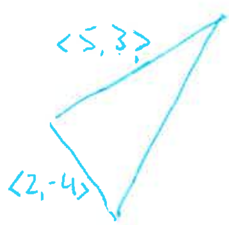






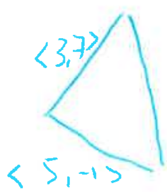
The area of the hexagon is given by the following

- Area of 1:



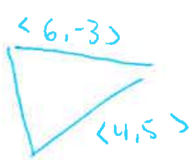
$\frac{1}{2}\left|\det\begin{pmatrix}5, 3\\ 2, -4\end{pmatrix}\right| = 13.$

- Area of 2



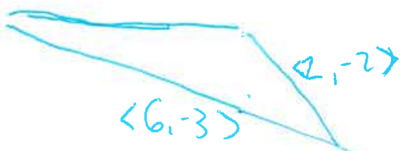
$\frac{1}{2}\left|\det\begin{pmatrix}5 & -1\\ 3 & 7\end{pmatrix}\right| = 19$

- Area of 3



$\frac{1}{2}\left|\det\begin{pmatrix}6 & -3\\ 4 & 5\end{pmatrix}\right| = 21$

- Area of 4



$\frac{1}{2}\left|\det\begin{pmatrix}2 & -2\\ 6 & -3\end{pmatrix}\right| = 3$

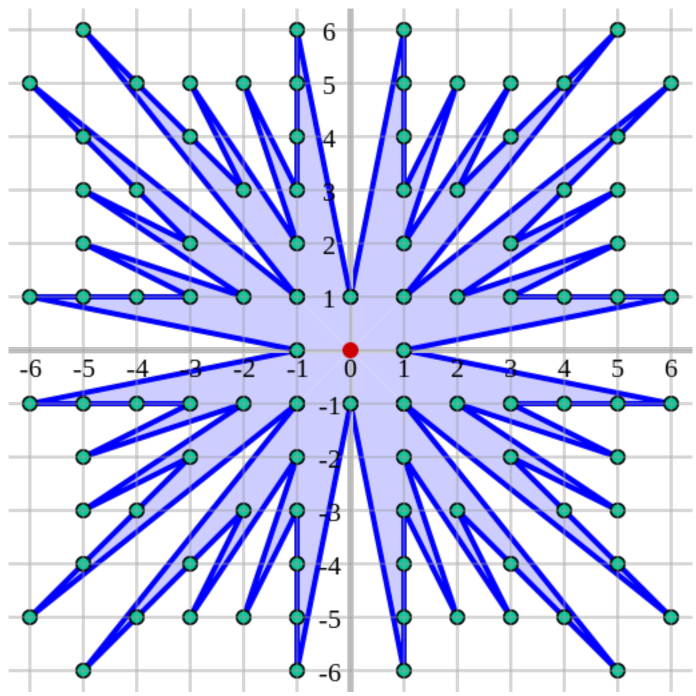
Total area: 56 square units.

Definition

A point $(v_1, v_2, \dots, v_d) \in \mathbb{R}^d$ is an integral point or a lattice point if $(v_1, \dots, v_d) \in \mathbb{Z}^d$.

A polytope is a lattice polytope if it is the convex hull of lattice points.

Pick's theorem



Picture: CMG Lee on Wikipedia

Goal: Counting (easily) the volume of lattice polytopes.

Theorem (Pick, 1899)

Let P be any lattice polygon, convex or not.

The area of P is given by

$$A = I + \frac{1}{2} B - 1,$$

where I is the number of lattice points in the interior of P, and B is the number of lattice points on the boundary.

Examples

Hexagon from previous page

$I = 53$

$B = 8$

$I + \frac{1}{2} B - 1 = 56$

Area = 56 square units.

Farey sunburst

$I = 1$

$B = 96$

$I + \frac{1}{2} B - 1 = 48$

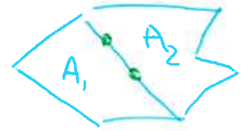
Area = 48 square units.

Proof

Overview of the proof:

① We prove the additive property, namely that if we partition a polygon into two polygons, we can sum the areas of the two polygons:

$$A = A_1 + A_2 = (I_1 + \tfrac{1}{2}B_1 - 1) + (I_2 + \tfrac{1}{2}B_2 - 1)$$



② We prove Pick's formula for all triangles.

(2a) We prove it holds for all rectangles with sides parallel to the axes.

(2b) We prove it holds for right triangles, with sides next to the right angles parallel to the axes.

(2c) We express any triangle as the difference of rectangles and right triangles with sides parallel to the axes.

③ Since we can triangulate any polygon, we conclude that Pick's theorem works for any polygon!

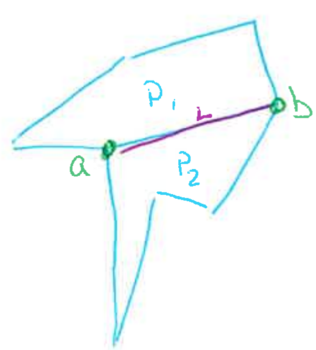
① Additive property.

Let P be a polygon with at least 4 vertices.

Break P into P_1 and P_2 by adding an edge between two non-adjacent vertices of P.

Let I_1, I_2 be the number of interior points of P_1, P_2, and B_1, B_2 be the number of boundary points.

Let L be the number of vertices on the new edge.



Then, $I = I_1 + I_2 + \underbrace{(L-2)}_{\text{exclude } a,b}$

$$B = (B_1 - L + 2) + (B_2 - L + 2) - 2$$

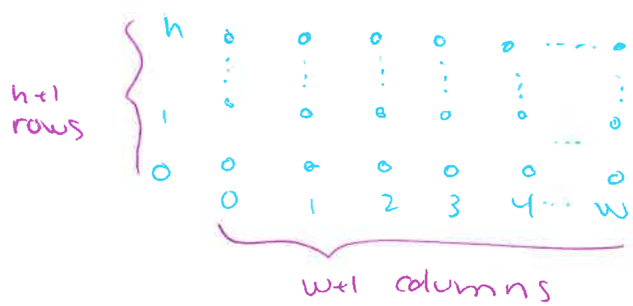
$$I + \tfrac{1}{2}B - 1 = I_1 + I_2 + \cancel{L} - 2 + \tfrac{1}{2}(B_1 + B_2 - \cancel{2L} + 2) - 1$$

$$= (I_1 + \tfrac{1}{2}B_1 - 1) + (I_2 + \tfrac{1}{2}B_2 - 1)$$

$$= A_1 + A_2.$$

This proves the additive property.

(2). (2a) A rectangle with sides parallel to the axes has a width, w, and a height, h. It looks like



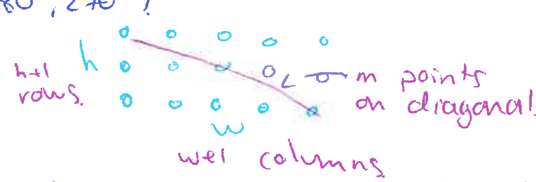
$I = (w-1)(h-1)$, $B = 2w + 2h$. Area is wh.

Using Pick's formula:

$$I + \tfrac{1}{2}B - 1 = wh - w - h + 1 + \tfrac{1}{2}(2w + 2h) - 1$$

$$= wh.$$

(2b). A right triangle with small sides parallel to the axes look like (up to rotation by $90°, 180°, 270°$)



Let m be the number of lattice points on the hypothenuse.

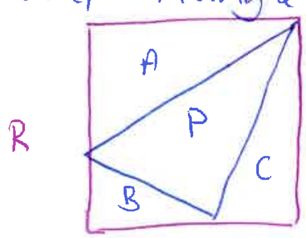
$$I = \frac{(h-1)(w-1) - (m-2)}{2}, \quad B = h + w + m - 1$$

÷2, since we partition the rectangle into two equal triangles

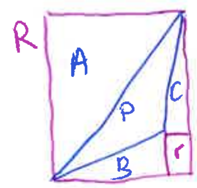
$$I + \tfrac{1}{2}B - 1 = \frac{hw - h - w - m + 3}{2} + \frac{h + w + m - 1}{2} - 1$$

$$= \frac{hw}{2} + 1 - 1 = \frac{hw}{2}.$$

(2c) Every triangle can be expressed as the difference of (2a) - (2b).



$P = R - A - B - C.$



$P = R - r - A - B - C.$

Because of the additive property, Pick's formula holds for all triangles. By triangulations, it holds for any polygon!

Dimension 3

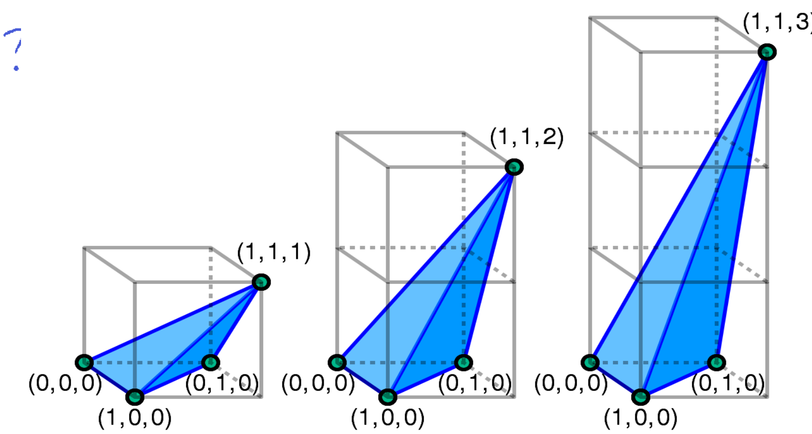
Can we count lattice points to find the volume of polyhedra?

Example (Reeve tetrahedra)

Consider the tetrahedron with lattice points $(0,1,0), (1,0,0)$, $(0,0,0)$ and $(1,1,n)$.

volume: $n/6$ (base · height / 3)

$I = 0, \quad B = 4.$ This is not possible!



Picture: CMG Lee on Wikipedia

Reference: [CCD], §2.6.