CATEGORIES AND FUNCTORS

DENIS AUROUX (BASED ON NOTES BY JOE HARRIS)

Many otherwise disparate fields of mathematics share some fundamental structures. The language of categories is a way to systematize these common structures; while it won't answer the questions arising in a particular area of mathematics, it may help suggest useful ways to think about them by analogy with other situations.

1. Categories

Definition 1.1. A *category* $\mathcal{C}$ consists of the following:

(1) a collection of *objects*, denoted $Ob(\mathcal{C})$;
(2) for every pair of objects $A, B \in Ob(\mathcal{C})$, a collection of *morphisms* from A to B, denoted $Mor(A, B)$; and
(3) a rule of *composition*: for every triple $A, B, C \in Ob(\mathcal{C})$, a map

$$Mor(A, B) \times Mor(B, C) \;\to\; Mor(A, C).$$

If $\phi \in Mor(A, B)$ and $\eta \in Mor(B, C)$, the image of (ϕ, η) is denoted $\eta \circ \phi$.

These have to satisfy some basic axioms:

(1) *associativity*: if $A, B, C, D \in Ob(\mathcal{C})$ are objects, and $\alpha \in Mor(A, B)$, $\beta \in Mor(B, C)$ and $\gamma \in Mor(C, D)$, then

$$\gamma \circ (\beta \circ \alpha) \;=\; (\gamma \circ \beta) \circ \alpha.$$

(2) *identity*: for every $A \in Ob(\mathcal{C})$, there exists an element $id_A \in Mor(A, A)$ such that for any $A, B \in Ob(\mathcal{C})$ and $f \in Mor(A, B)$,

$$f \circ id_A \;= f \;=\; id_B \circ f.$$

The key to understanding categories is to look at examples, to see how the sort of constructions you're familiar with in various settings all fit into this framework.

Example 1.2 (Sets). In this category, the objects are sets; the morphisms $Mor(A, B)$ are simply maps $f : A \to B$ of sets, and composition is, well, composition. Some variants:

- The category of *finite sets*: objects are finite sets, and morphisms are maps of sets.
- We can define a category of *pointed sets*: the objects are pairs (A, x) with A a set and $x \in A$ an element; the morphisms $Mor((A, x), (B, y))$ are maps $f : A \to B$ such that $f(x) = y$.

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Example 1.3 (Groups). The objects are all groups; the morphisms $Mor(A, B)$ will be group homomorphisms.

Variants: we can consider only finite groups, or only abelian groups, or only finite abelian groups, with the morphisms again being group homomorphisms. We can also define a category whose objects are rings, and whose morphisms are ring homomorphisms.

Example 1.4 (Finite-dimensional vector spaces over K). Fix a field K. We can then form a category $Vect_K$ whose objects are finite-dimensional vector spaces over K, and whose morphisms are simply linear maps.

Example 1.5 (Topological spaces). The objects are topological spaces, and the morphisms are continuous maps. As a variant, we can define the category of *pointed topological spaces*, whose objects are pairs (A, p) with A a topological space and $p \in A$ a point of A; morphisms $Mor((A, p), (B, q))$ are continuous maps $f : A \to B$ such that $f(p) = q$.

As you can imagine, there are many more examples: algebraic varieties over a field K; schemes; differentiable manifolds; Lie groups; Lie algebras, and so on.

One remark we should make here: in all the examples we've mentioned, the objects are "decorated sets:" sets with some additional structure, such as a topology, or a group law, or just a distinguished element; and the morphisms are set maps that respect this structure. This doesn't have to be the case for an arbitrary category, though it is true of most of the ones we deal with in practice.

2. Isomorphisms

Fix a category $\mathcal{C}$, and consider two objects $A, B \in Ob(\mathcal{C})$.

Definition 2.1. A morphism $f \in Mor(A, B)$ is an *isomorphism* if there exists $g \in Mor(B, A)$ such that $g \circ f = id_A$ and $f \circ g = id_B$.

The inverse g, if it exists, is unique, and is then denoted f^{-1}.

Proposition 2.2. *For any object $A \in Ob(\mathcal{C})$, the set of* automorphisms *of A, $Aut(A) = \{f \in Mor(A, A) \,|\, f \text{ is an isomorphism}\}$, equipped with composition, is a group.*

Proof. The composition of two isomorphisms f, g is an isomorphism, as one checks that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, so composition is a well-defined operation on $Aut(A)$. Observe that id_A is an automorphism ($id_A^{-1} = id_A$), and acts as identity for composition. Moreover, the inverse of an isomorphism is an isomorphism (the inverse of f^{-1} is f) Associativity follows from the axioms of a category. □

Example 2.3. In the category of finite sets, objects are isomorphic if and only if they have the same cardinality; if $|A| = n$ then $Aut(A)$ is the group of permutations of A, isomorphic to the symmetric group S_n.

In the category of finite dimensional vector spaces over K, two objects are isomorphic if and only if they have the same dimension; and $Aut(V)$ is the group of invertible linear operators on V (isomorphic to the group $GL_n(K)$ of invertible $n \times n$ matrices with entries in K, where $n = \dim V$).

Exercise 2.4. Show that, if $A, B \in Ob(\mathcal{C})$ are isomorphic in $\mathcal{C}$, then the groups $Aut(A)$ and $Aut(B)$ are isomorphic in the category of groups. More precisely, given an isomorphism $f \in Mor(A, B)$, show that conjugation by f, i.e. mapping $g \in Aut(A)$ to $c_f(g) = f \circ g \circ f^{-1} \in Aut(B)$, defines a group isomorphism.

A category with a single object, and where all morphisms are isomorphisms, is essentially the same thing as a group (namely, the group of automorphisms of the only object). This generalizes to the following notion:

Definition 2.5. A *groupoid* is a category in which all morphisms are isomorphisms.

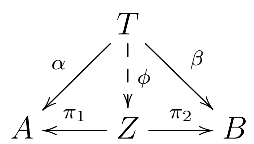
The main difference with a group is that morphisms can't always be multiplied (composed): for $f \in Mor(A, B)$ and $g \in Mor(C, D)$, the composition $g \circ f$ is only defined if $B = C$; however, when defined, composition is associative; there is an identity element for each object; and inverses exist.

Example 2.6. The category whose objects are finite sets and morphisms $Mor(A, B)$ are only *bijections* from A to B (rather than all maps) is a groupoid. So is the category whose objects are groups and morphisms are group isomorphisms; or the category in which objects are topological spaces and morphisms are homeomorphisms.

3. Products and sums

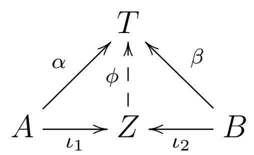
Familiar constructions that we encounter in a wide range of settings are often special cases of general constructions in category theory. We will mention just two here, by way of examples: products and sums.

Definition 3.1. Let $\mathcal{C}$ be a category and $A, B \in Ob(\mathcal{C})$ objects. By a *product* $A \times B$ we will mean an object $Z \in Ob(\mathcal{C})$ together with morphisms $\pi_1 : Z \to A$ and $\pi_2 : Z \to B$, with the property that for any object $T \in Ob(\mathcal{C})$ and morphisms $\alpha : T \to A$ and $\beta : T \to B$, there exists a unique map $\phi : T \to Z$ such that $\alpha = \pi_1 \circ \phi$ and $\beta = \pi_2 \circ \phi$.



We have a similar definition of sum:

Definition 3.2. Let $\mathcal{C}$ be a category and $A, B \in Ob(\mathcal{C})$ objects. By a *sum* $A + B$ we will mean an object $Z \in Ob(\mathcal{C})$ together with morphisms $\iota_1 : A \to Z$ and $\iota_2 : B \to Z$, with the property that for any object $T \in Ob(\mathcal{C})$ and morphisms $\alpha : A \to T$ and $\beta : B \to T$, there exists a unique map $\phi : Z \to T$ such that $\alpha = \phi \circ \iota_1$ and $\beta = \phi \circ \iota_2$.



Three remarks are in order. First: *products and sums are unique* (up to isomorphism), *if they exist.* Second, they may not necessarily exist in a given category; one of the first things you check when working in a new category is whether they do.

Lastly, a general remark: like many definitions in category theory, it takes a while to unpack the terms. But, once you have, you realize that this is a construction central to many disparate areas of mathematics: products exist in the categories of finite sets, groups, topological spaces and many more. This is, in a sense, the point of category theory: to understand common themes in different settings.

Exercise 3.3. Verify that, in the category of sets, products are Cartesian products $A \times B$ and sums are disjoint unions $A \sqcup B$.

Exercise 3.4. Verify that, in the category of groups, products are Cartesian products $G \times H$. Sums do exist in the category of groups, but are given by a more complicated construction called the *free product*; whereas in the category of abelian groups, they are again Cartesian products (also known as direct sums). Verify that the sum of $\mathbb{Z}$ and $\mathbb{Z}$ in the category of abelian groups is $\mathbb{Z}^2$, while in the category of all groups it is the free group on two generators.

Exercise 3.5. Verify that, in the category $Vect_K$ of finite dimensional vector spaces over the field K, sums and products are both direct sums. What happens if we consider the category of all vector spaces over K (not necessarily finite-dimensional) and extend the notions to products and sums of infinitely many objects?

4. Functors

Inevitably, we want to find constructions that map things from one category to another. The key notion for doing so is that of *functor.* As with everything else, the definition takes a little unwinding, but it is worth the effort.

Definition 4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is the data of:

(1) a map $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$, associating to every object $A \in Ob(\mathcal{C})$ an object $F(A) \in Ob(\mathcal{D})$; and
(2) for every pair of objects $A, B \in Ob(\mathcal{C})$, a map
$$Mor_{\mathcal{C}}(A, B) \ \to \ Mor_{\mathcal{D}}(F(A), F(B))$$
associating to every morphism $\phi : A \to B$ a morphism $F(\phi) : F(A) \to F(B)$.

These have to respect the law of composition: $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$, and the identity: $F(id_A) = id_{F(A)}$.

A *contravariant functor* is defined similarly, except the direction of morphisms is reversed, so that $F : \mathcal{C} \to \mathcal{D}$ associates to a morphism $\phi : A \to B$ in $\mathcal{C}$ a morphism $F(\phi) : F(B) \to F(A)$ (and we now require that $F(\alpha \circ \beta) = F(\beta) \circ F(\alpha)$).

There are many examples. The simplest ones are *forgetful functors*: if the objects in a category $\mathcal{C}$ are decorated sets—e.g., groups, rings, vector spaces, topological spaces, etc.—we can simply forget the extra structure and associate to every object $A \in Ob(\mathcal{C})$ its underlying set; this defines a functor $F : \mathcal{C} \to (Sets)$.

Less trivial examples abound. For example, the *fundamental group* is a functor from the category of pointed topological spaces to the category of groups, and similarly homology groups are functors from the category of topological spaces to the category of abelian groups (cohomology groups are contravariant functors between these categories).

Exercise 4.2. Fix a field K. If V is a vector space over K, we define the *dual space* V^* to be the vector space $\mathrm{Hom}(V, K)$ of linear maps from V to K. Show that this is a contravariant functor from the category $Vect_K$ of finite-dimensional vector spaces over K to itself, and that its square is the identity functor.