Your name: Solutions

Section (please tick):

☐ Ma (BL)  ☐ Ma (C)  ☐ Lafreniere (D)  ☐ Kulkarni (E)  ☐ Lin (F)

INSTRUCTIONS

You may begin the exam when ready.

Write your name in the space provided above, and check one box to indicate which section of the course you belong to.

Unless otherwise stated, you must justify your solutions to receive full credit. Work that is illegible may not be graded. Work that is scratched out will not be graded.

It is fine to leave you answer in a form such as \( \ln(0.02) \) or \( \sqrt{123412} \) or \( (1341)^4(1231)^{-1} \). However, if an expression can be easily simplified (such as \( e^{\ln(0.02)} \) or \( \cos \pi \)), you should simplify it.

The Honor Principle requires that you neither give nor receive any aid on this exam. This exam is open book (you may use only the textbook, notes, and videos from canvas). You may not use calculators, software, or the internet outside of canvas. You may ask the instructors for clarification on problems through e-mail.

The exam has been created with the intended length of 2 hours. You must submit your solutions through Canvas by December 3rd, 2:30pm EST. Directions to submit the exam will be on Canvas.

Good luck!

Honor statement: I have neither given nor received any help on this exam, and I attest that all of the answers are my own work.

Signature: __________________________
Multiple Choice Questions

For problems 1-8, no justification is required.

(1) Every function is differentiable over its domain.
   (a) True  
   (b) False  
   \[ e.g. \; |x| = y \; \text{is not differentiable at} \; x = 0, \]
   \[ \text{non-continuous functions are not differentiable}. \]

(2) True or False: Let \( f(x) \) be an odd integrable function. Then
   \[ \int_{-1}^{4} f(x)dx = \int_{1}^{4} f(x)dx. \]
   (a) True  
   (b) False  
   \[ \text{For odd functions,} \; \int_{-a}^{a} f(x) dx = 0. \]
   \[ \text{Since} \; \int_{-1}^{1} f(x)dx = \int_{1}^{4} f(x)dx + \int_{1}^{1} f(x)dy, \text{the statement is true}. \]

(3) Given an \( \epsilon > 0 \), what should we choose our integer \( N \) to be in order for the sequence
   \[ a_n = \frac{5}{n^3} \]
   to converge to 0?
   (a) \( N \) greater than \( \frac{5}{\epsilon^3} \).
   (b) \( N \) greater than \( \frac{3}{\sqrt[3]{\epsilon}} \).
   (c) \( N = 100. \)
   \[ \text{So} \; N \; \text{must be greater than} \; \frac{3}{\sqrt[3]{\epsilon}}. \]

(4) Suppose \( f(x) = 1 + \frac{1}{x} \) and \( g(x) = \frac{2}{x(x - 2)} \) then \( \lim_{x \to 0} (f(x) + g(x)) \) is
   (a) 0.
   (b) \( \frac{1}{2} \).
   (c) Undefined or does not exist.
   (d) \( \infty. \)
   \[ \lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} \frac{1 + \frac{2}{x}}{x(x - 2)} = \lim_{x \to 0} \frac{1}{x} + \frac{2}{x(x - 2)} = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}. \]

(5) Let \( f(x) = \sin(3x) \cos\left(\frac{1}{5x}\right) \), then \( \lim_{x \to 0} f(x) \)
   (a) does not exist.
   (b) exists and equals 0.
   (c) exists and equals \( \frac{3}{5} \).
   \[ \text{Squeeze Theorem:} -1 \leq \cos\left(\frac{1}{5x}\right) \leq 1. \; \text{Then} \]
   \[ \lim_{x \to 0} \sin(3x) - 1 \leq \lim_{x \to 0} \sin(3x) \cos\left(\frac{1}{5x}\right) \leq \lim_{x \to 0} \sin(3x) - 1. \]
   \[ \text{Because} \; \sin(3x) \; \text{is continuous at} \; x = 0, \; \lim_{x \to 0} \sin(3x) = \sin(0) = 0. \]
   \[ \text{Thus,} \; 0 \leq \lim_{x \to 0} \sin(3x) \cos\left(\frac{1}{5x}\right) \leq 0 \; \Rightarrow \lim_{x \to 0} \sin(3x) = 0. \]
   \[ \text{and} \; \lim_{x \to 0} \sin(3x) \cos\left(\frac{1}{5x}\right) = 0. \]
(6) Let \( f(x) \) and \( g(x) \) be two differentiable functions. If \( f'(x) = g'(x) \) for all values of \( x \), then \( f(x) = g(x) + C \) for all values of \( x \).

(a) True
(b) False

(7) From the following options
- I. The limit is not an indeterminate form.
- II. At least one of the functions is not differentiable over an open interval containing 0 (except possibly at 0).
- III. L'Hôpital's rule can be applied.

Apply the label that matches the following limits:

(a) \( \lim_{x \to 0} \frac{\tan(x)^2 \tan\left(\frac{1}{x}\right)^2}{x^{10}} \quad \text{II. } \tan\left(\frac{1}{x}\right) \text{ is not differentiable around } a \quad (\lim_{x \to 0} \tan\left(\frac{1}{x}\right) \text{ does not exist}) \)

(b) \( \lim_{x \to 0} \frac{x^{100}}{e^x} \quad \text{I. } \text{Numerator is 0, Denominator is 1} \)

(c) \( \lim_{x \to 0} \frac{1}{\sin x} \quad \text{III. of the form } \frac{\infty}{0}, \text{ and } \frac{1}{x} \text{ and } \frac{1}{\sin(x)} \text{ are differentiable} \)

(8) \( \int_a^b \ln(3t) \, dt = \int_c^d s \ln(z) \, dz \). Express \( c \) and \( d \) in terms of \( a \) and \( b \) and \( s \) as a real number (no justification required).

(\text{substitution})

Set \( z = 3t \). Then, \( t = \frac{z}{3} \), \( dt = \frac{dz}{3} \).

When \( a = t \), \( z = 3a \), and when \( b = t \), \( z = 3b \).

Then,

\[ \int_{3a}^{3b} \frac{\ln(z)}{3} \, dz = \int_{3a}^{3b} \frac{1}{3} \ln(z) \, dz \]

\( c = 3a \), \( d = 3b \), \( s = \frac{1}{3} \).
Long Answer Questions

For problems 9-14, please show all your work and explain your reasoning to receive full credit.

(9) (10 points) Let \( f(x) \) be a differentiable function such that

\[
\int_0^{\pi/2} f'(x) \sin(x) \, dx = 3
\]

and

\[
\int_0^{\pi/2} f(x) \cos(x) \, dx = 5,
\]

find \( f(\pi/2) \).

Using integration by parts:

\[
\int u \, dv = uv - \int v \, du
\]

Set:

\[
\begin{align*}
    u &= f(x) & v &= \sin(x) \\
    du &= f'(x) \, dx & dv &= \cos(x) \, dx
\end{align*}
\]

Hence,

\[
\int_0^{\pi/2} f(x) \cos(x) \, dx = f(x) \sin(x) \bigg|_0^{\pi/2} - \int_0^{\pi/2} \sin(x) f'(x) \, dx.
\]

By hypothesis, we know the values of

\[
\int_0^{\pi/2} f(x) \cos(x) \, dx = 5 \quad \text{and} \quad \int_0^{\pi/2} \sin(x) f'(x) \, dx = 3.
\]

So

\[
5 = \left( f(\pi/2) \sin(\pi/2) - f(0) \sin(0) \right) - 3,
\]

which means

\[ f \left( \frac{\pi}{2} \right) = 8.\]
(10) (12 points) Let
\[ g(x) = \int_{-1}^{e^{2x}} \cos(t) \sqrt{4 - t^2} dt. \]

(a) What is the domain of \( g(x) \)?

- \( \cos(t) \sqrt{4 - t^2} \) is only defined over \([-2, 2]\), since \( 4 - t^2 \) has to be nonnegative.
- Over \([-2, 2]\), \( \cos(t) \sqrt{4 - t^2} \) is continuous.
- Continuous functions are integrable.
- So \( -2 \leq e^{2x} \leq 2 \). However, for all \( x \), \( e^{2x} \geq 0 \).
  
  For \( e^{2x} \leq 2 \), that means \( 2x \leq \ln(2) \), so \( x \leq \frac{\ln(2)}{2} \).
  
  Hence, the domain of \( x \) is \( (-\infty, \frac{\ln(2)}{2}] \).

(b) What is \( g'(0) \)?

By Fundamental Theorem of Calculus (part 1), and since \( \cos(t) \sqrt{4 - t^2} \) is continuous over \([-1, 0]\),

\[
g'(x) = \cos(e^{2x}) \sqrt{4 - e^{4x}} \cdot (e^{2x})'
\]

\[ = 2e^{2x} \cos(e^{2x}) \sqrt{4 - e^{4x}} \]

At \( x = 0 \):

\[ g'(0) = 2e^{2\cdot0} \cos(e^{2\cdot0}) \sqrt{4 - e^{4\cdot0}} \]

\[ = 2 \cdot 1 \cdot \cos(1) \sqrt{4 - 1} \]

\[ = 2 \sqrt{3} \cos(1). \]
(11) (12 points) Given the following graph, and the region bounded by the $x$-axis, the function $y = \frac{4}{\pi}x - 1$ and the function $y = \sin(x)$. Find the area of the region in the graph. Note that the graphs intersect at $x = \pi/2$.

\[\begin{align*}
\text{At } x = \frac{\pi}{2}, \quad \sin(x) &= 1 \text{ and } \frac{4}{\pi}x - 1 = 2 - 1 = 1. \\
\text{This area is the area under } y = \sin(x) \text{ between } 0 \text{ and } \frac{\pi}{2}, \text{ minus the area under } y = \frac{4}{\pi}x - 1 \text{ between its } x \text{-intercept and } \frac{\pi}{2}. \text{ Its } x \text{-intercept is} \\
0 = \frac{4}{\pi}x - 1 \implies 1 = \frac{4}{\pi}x \implies x = \frac{\pi}{4}. \\
\text{Since the area under a curve is the definition of a definite integral, the area we are looking for is} \\
\int_0^{\frac{\pi}{2}} \sin(x) \, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4}{\pi}x - 1 \, dx = \left[ -\cos(x) \right]_0^{\frac{\pi}{2}} - \left[ \frac{2x^2}{\pi} - x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
= -\cos(\frac{\pi}{2}) + \cos(0) - \left( \frac{2\pi^2}{4\pi - \frac{\pi^2}{2}} \right) \\
= 1 - \frac{\pi^2}{8} - \frac{\pi^2}{2} = 1 - \frac{\pi^2}{8}.
\end{align*}\]
(12) (10 points) Let \( f(x) \) be a continuous function on [0, 10]. From FTC part 1, \( F(x) = \int_0^x f(t)dt \) is a differentiable and continuous function on [0, 10]. If \( \int_0^{10} f(t)dt = 10 \), show that there is a real number \( 0 \leq c \leq 10 \) such that \( \int_0^c f(t)dt = \pi \).

**Intermediate Value Theorem (IVT):**

If \( g(x) \) is a continuous function over \([a, b]\), then for any \( d \in [g(a), g(b)] \), there exists a \( c \in [a, b] \) such that \( g(c) = d \).

Since \( F(x) \) is continuous over \([0, 10]\), we satisfy the hypothesis:

\[ F(0) = \int_0^0 f(t)dt = 0 \]

As given by the problem:

\[ F(10) = (0) \]

So for any number \( d \in [0, 10] \), there exists \( c \in [0, 10] \) such that \( F(c) = d \).

Since \( 10 \) is in \([0, 10]\), IVT shows there exists a real number \( c \in [0, 10] \) such that

\[ \pi = F(c) = \int_0^c f(t)dt. \]
(13) (18 points) Evaluate
\[ \lim_{n \to \infty} \sum_{i=1}^{n} \ln\left(\frac{3i}{n} + 2\right) \frac{3}{n}. \]

Make sure you provide all justification.

The sum \[ \sum_{i=1}^{n} \ln\left(\frac{3i}{n} + 2\right) \frac{3}{n} \] is a Riemann sum; taking its limit as \( n \to \infty \) gives a definite integral.

1. Decompose the Riemann sum as \[ \sum_{i=1}^{n} f(x_i^*) \Delta x \]
   
   Let \( \frac{3}{n} \Delta x \). Then \( \frac{3i}{n} + 2 = \Delta x + a \), and \( a = 2 \) is the lower bound of the integral. The whole interval measures \( n \cdot \Delta x = 3 \), so \( b = 5 \) is the upper bound of the integral.

   The function is \( \ln(x) : [a, b] \), so \[
   \sum_{i=1}^{n} \ln\left(\frac{3i}{n} + 2\right) \frac{3}{n},
   \]

   and

   \[ \lim_{n \to \infty} \sum_{i=1}^{n} \ln(x_i^*) \Delta x = \int_{2}^{5} \ln(x) \, dx. \]

2. To integrate \( \ln(x) \), we use integration by parts:

   \[ \int_{2}^{5} \ln(x) \, dx = \left[ x \ln(x) \right]_{2}^{5} - \int_{2}^{5} \frac{x}{x} \, dx \]

   \[ = (x \ln(x)) \bigg|_{2}^{5} - \int_{2}^{5} \frac{x}{x} \, dx \]

   \[ = (x \ln(x)) \bigg|_{2}^{5} - \int_{2}^{5} 1 \, dx \]

   \[ = 5 \ln(5) - 2 \ln(2) - 2 \]

   \[ = 5 \ln(5) - 2 \ln(2) - 3. \]

   Hence,

   \[ \lim_{n \to \infty} \sum_{i=1}^{n} \ln\left(\frac{3i}{n} + 2\right) \frac{3}{n} = 5 \ln(5) - 2 \ln(2) - 3. \]
(14) (14 points) Two trains are moving along parallel tracks that are 20 km apart, one heading east and the other heading west. Assuming that each train moves at a constant speed 55 km/hr, find the rate at which the distance between the trains is changing when the distance is 25 km apart, heading toward each other.

![Diagram showing two trains moving towards each other with distances and speeds labeled.]

Let \( t \) be time, in hours. 
\( D \) be the distance between the trains. 
At \( t = 0 \), \( D = 25 \), and the trains are at milepoints that are \( \sqrt{25^2 - 20^2} = 15 \) km apart.

The milepoint distance between the two points is shrinking at a rate of \( 2 \times 55 \text{ km/hr} = 110 \text{ km/hr} \).

we are looking for the rate at which the distance changes at \( t = 0 \). This is \( \frac{dD}{dt} \) at \( t = 0 \):

Using implicit differentiation:

\[
2D \frac{dD}{dt} = 2 \left( 15 - 110t \right) \cdot (-110)
\]

\[
\Rightarrow \frac{dD}{dt} = \frac{-220 \left( 15 - 110t \right)}{2D}
\]

At \( t = 0 \):

\[
\frac{dD}{dt} \bigg|_{t=0} = \frac{-110 \cdot 15}{25} = -66
\]

The trains are getting closer at a rate of 66 km/h.