\[ \rightarrow \text{ BBGKY to Boltzmann} \]

\[ + \# \text{ Thm.} \]

- \( N \) particle Hamiltonian, \( N \gg 1 \)

system described by:

\[ f(t, x_j, p_j; x_j^0, p_j^0 \cdots x_j^0, p_j^0) \]

\( \rightarrow N \) particle distribution

This satisfies full Liouville Equation
\[ \frac{dP^x_i}{dt} + \sum_{j=1}^{N} \left\{ \mathbf{v}_i \cdot (\mathbf{x}_j - \mathbf{x}_i) \right\} = \dot{\mathbf{p}}_i \cdot \left( \mathbf{v}_j - \mathbf{v}_i \right) \]

\[ \mathbf{v}_i = \mathbf{v}_i \]

\[ \mathbf{P}_i = -\frac{\hbar}{2} \sum_{j \neq i} \frac{\mathbf{v}_j \cdot \partial}{\partial x_i} \frac{\partial}{\partial x_i} \]

**Lennard-Jones Thm:** $P^N$ continued along $N$ particle orbits

$P^N \rightarrow$ exactly useless

Seek: $P^1, P^2 \rightarrow$ self for a particle

$F(x, v, t) \rightarrow$ phase space density

Approach: Integrate out additional particles $\rightarrow$ reduce description

Catch: Basic interaction is 2-body
\( V_{ij} \) is 2-particle interaction \( \Rightarrow \) necessary only enters with \( f_2 \).

Need treat all possible pairs \( \Rightarrow \)

\[
\mathfrak{A} + v_i \cdot \mathfrak{A} f_i^2
\]

\[
= (N-1) \int d \Gamma_z \frac{\partial \mathbf{V}_{z2}}{\partial x_i} \cdot \frac{\partial f_i}{\partial \mathbf{A}}
\]

\# binary pairs
N particles 2-particle interaction 2-body distribution

M.B.: \( \mathfrak{F}_{11} = \int (\cdot) f_2(\cdot) \)

\( \Rightarrow \) hierarchy problem
- how to couple?

Need \( f_2 \) eqn. 1

\( \Rightarrow \) how? - integrate out from 3 on.
\[
\begin{aligned}
\frac{df^1}{dt} + \frac{U \cdot \dot{f}^2}{dx_1} + \frac{U_2 \cdot \dot{f}^2}{dx_2} \\
- \frac{\nabla V_{x_2}}{x_2} \cdot \frac{df^1}{\partial \phi} - \frac{\nabla V_{x_2}}{x_2} \cdot \frac{df^2}{\partial \phi}
\end{aligned}
\]

\[
= (N-2) \int d^3 \left[ \frac{df^1}{\partial \eta_1} \frac{df^3}{\partial \eta_3} + \frac{df^3}{\partial \eta_1} \frac{df^2}{\partial \eta_2} \right] \\
\text{# triplets}
\]

Can we simplify this?

\[
\begin{aligned}
\frac{df^1}{dt} + U \cdot \dot{f}^2, \quad \frac{df^2}{dt} = \frac{\nabla V_{x_2}}{x_2} \cdot \frac{df^2}{\partial \phi}
\end{aligned}
\]

\[
= (N-2) \int d^3 \left[ \frac{df^1}{\partial \eta_1} \frac{df^3}{\partial \eta_3} + \frac{df^3}{\partial \eta_1} \frac{df^2}{\partial \eta_2} \right] \\
\text{Look at } \frac{2}{1}
\]

\[
\rightarrow \text{ exploit low volume filling of interaction } \quad \pi d^2 \ll 1.
\]
\[ 2 \sim N \int d^3 \vec{x} \int dA \frac{df_2}{dA} \cdot \frac{dV_3}{dA} \]

ign \( N \rightarrow \) large \( N \)

\[ f(2) \sim \frac{1}{(AP)^3} \frac{1}{\text{Vol.}} \]

normalization

\[ \Rightarrow \int dP_3 \text{ cancels } \sqrt{AP^3} \text{ normalization} \]

\[ \sqrt{d^3} \text{Volume} \]

\[ 2 \sim \frac{d^3 N}{\text{Vol.}} \left( \frac{dV_3}{dA} \right) \left( \frac{df}{dA} \right) \]

\[ \sim n d^3 \left( \frac{dV_3}{dA} \right) \left( \frac{df}{dA} \right) \]
\[ \frac{df}{dt} \left( t_i, \Gamma_i, \Gamma_2 \right) = 0 \]

- constitutes truncation of BBSKY hierarchy, for dilute gas
- key is \( d^3 / \Gamma^3 \ll 1 \)
  - has scale ordering

\[ \frac{df}{dt} = 0 \] is straightforward for dilute.

\[ \rightarrow \text{if posit statistical independence of colliding particles, etc.} \]
- Molecular areas.
\[ f(t, l_1, 2) = f(t, 1) \cdot f(l_2) \]

then,

\[ F(t, l_1, l_2) = F(t, l_1) \cdot F(l_2, \beta_2) \]

serves as i.e. for \[ \frac{dF(\beta_1)}{dt} \]

\[ \frac{dF(\beta_1)}{dt} = 0 \]

so \[ F(\beta_1) \] always factorizes

- consistent with "Freely moving particles, interacting only within \( d \ll \frac{1}{\gamma} \),"

- so \[ F(\beta_1) = F(l_1) \cdot F(l_2) \] and

\[ \frac{\partial F_0}{\partial \gamma} + \nu \cdot \delta F_0 = N \int dR_1 \frac{dV_2}{2} \cdot \delta \left[ F(\nu) \cdot F(\gamma) \right] \]

\[ \rightarrow \text{ Boltzmann Equation} \]
Boltzmann Eqn.

\[ \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = \frac{1}{\hbar^2} \int C(f) \, d\mathbf{p} \]

\[ C(f) = \int d\mathbf{p}_2 \, \frac{\partial^2}{\partial \mathbf{p}_1 \partial \mathbf{p}_2} \left[ f(\mathbf{p}_1 + \mathbf{v}, t) \right] \]

- Absorbed normalization
- \( C(f) \equiv \text{collision operator (integral)} \)
- \( C(f) \) nonlinear \( \rightarrow 2 \text{ body collision} \)
- have "test particle" (\( f(\mathbf{p}_1) \)) scattered by other "field particles"
  but "test field" the same
  \( \Rightarrow \text{nonlinearity} \)
- \( C(f) \sim V_{\text{coll}} \)

\[ C(f) \equiv -\gamma [\mathbf{F} - \mathbf{F}_{\text{eq}}] \quad \text{(Chod3)} \]
\[- \frac{df}{dt} = \mathcal{C}(\mathbf{f}) \]

Phase space density is conserved along particle orbits up to collisions.

- What if \( \mathcal{C}(\mathbf{f}) \to 0 \)

have:

\[- \frac{df}{dt} + \mathbf{v} \cdot \nabla f + \frac{f}{\mathcal{M}} \frac{\partial f}{\partial y} = 0 \]

\[ F = \int E \]

\[ \Delta E = \frac{U_{\text{JSov}}}{\text{Jeans}} \]

Equation:

\[ \Delta E = \frac{U_{\text{JSov}}}{\text{Jeans}} \]

\[ \text{Continuity eqn. for incompressible flow of phase space fluid} \]

- Plasma:

\[ \bar{\mathcal{L}} = \frac{\lambda_0 < \lambda_{\text{Debye}} < \lambda_{\text{mfp}} < L}{\text{long range/decays} \Rightarrow \text{Fokker-Planck}} \]
L2b - H Theorem.

→ To H Theorem!

→ Recall:
  - Hamiltonian system
  - Non-dissipative interactions
  - From Liouville (for $N \sim NA \gg 1$) to Boltzmann:
    key
    → key is diluteness ordering
    $d < \bar{n}^{-1/3} < \text{mfp} < L$
  - Molecular Chaos Assumption
    Detailed Balance
    $f_N \text{ eqn.} \rightarrow f_\text{eqn.}$

→ Boltzmann Equation.
\[ \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = C(f) \]

\[ C(f) = \int d\Omega_2 \frac{\partial}{\partial \Omega_2} \cdot \mathbf{V} \left[ f(\mathbf{r} + \mathbf{V} \Delta t) f(\mathbf{r} - \mathbf{V} \Delta t) \right] \]

Collision operators

\[ \text{Nonlinear} \]

In reality, every particle (dot) is both a test particle and a field particle.
2. H- Theorem:
   - What does it say/mean?
   - What makes it remarkable?
   - What does it rest upon? (Assumptions)
   - How to prove?
Onward:

- Can also write Boltzmann Equation as change in occupation of state $\Delta$.

\[ \frac{df(p)}{dt} = \text{rate of change of } f \text{ due to collisions (interactions)} \]

\[ \frac{df(p)}{dt} = \text{rate scatttering in} - \text{rate scattering out} \]

(anticipates Master Equation)

\[ \text{in, out?} : \]

\[ \begin{array}{c}
\text{in} \\
\text{interaction} \\
p_1 \\
p_1' \\
p \\
p' \\
\text{out}
\end{array} \]

\[ = \text{in} \]
\[ \text{out} = \int d\mathbf{P}' \int d\mathbf{P}_1 \int d\mathbf{P}_1' \ f(\mathbf{P}') f(\mathbf{P}_1') w(\mathbf{P}, \mathbf{P}_1, \mathbf{P}_1', \mathbf{P}') \]

\[ \text{as} \quad w = w^T \quad \text{(reversible micro-dynamics)} \]

\[ \frac{df(\mathbf{P})}{dt} = \int d\mathbf{P}_1 \int d\mathbf{P}' \int d\mathbf{P}_1' \ w(\mathbf{P}, \mathbf{P}_1, \mathbf{P}_1', \mathbf{P}') \times \left( f(\mathbf{P}') f(\mathbf{P}_1') - f(\mathbf{P}) f(\mathbf{P}_1) \right) \]

N.B. - \( \mathbf{P} + \mathbf{P}_1 = \mathbf{P}' + \mathbf{P}_1' \)

so detailed balance applies.

\[ \mathbf{F} = -\mathbf{F}_0 \quad \mathbf{M} = \mathbf{c} \exp \left[ -\frac{\varepsilon + \mathbf{P} \cdot \mathbf{V}}{T} \right] \]
So \( C(f_0) = 0 \) conservation/energy/momentum

and will show Maxwellian renders \( \frac{dS}{dt} = 0 \).

N.B. - What about \( V_{1/2}(x) \) ?

\[ \Rightarrow \text{Stosszahlansatz} \]

L. Boltzmann

Total \# of \( (u, v) \) collisions taking place in \( dt \)

\[ = \left( \text{Volume of } u, v \text{ collision cylinder} \right) \\
\times \left( \# \text{ of particles with } u_1 \text{ per volume} \right) \]

\[ \int_{u_1 - v}^{u_1 + v} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{F}(u_1) \, dv \, dv_1 \, du_1 \]

\[ \Gamma \approx d^2 \]
and:

- dilute: non-overlapping cylinders
- collisions as 'point events' ✗

\[ t_1 < \tau < t_{\text{merge}} \quad \text{ordering} \]

Also:

\[
\left[ V_{12} \to \rho C_i - v_2 \right] \quad \text{integrates:}
\]

\[- W d^3v_1 d^3v_2 = \text{valid} \]

relates transition (probability) to familiar items like cross-section.

\[ t_{\text{merge}} \quad t_{\text{merge}} = \frac{1}{\sqrt{\rho}} \]

Onward: \[ \Rightarrow H - \text{Theorem.} \]

- a gas left alone will evolve to an equilibrium of maximal entropy.

- evolution accompanied by entropy production
\[ \frac{dS}{dt} \geq 0 \]

- **Evolution is to uniform Maxwellian**

- Ideal gas:

\[ S = \int dx \int dp \ f \ln(e/T) \]

\[ = \int dx \int dp \ f \ln f \]

\[ \frac{dS}{dt} = -\int d\Omega \left[ \frac{df}{dt} \ln f + f \frac{df}{dT} \right] \]

\[ = -\int d\Omega \left[ c(T) \ln f + c(f) \right] \]

\[ \int d\Omega \ c(f) = 0 \quad \text{show} \]

\[ \frac{dS}{dt} = -\int d\Omega \ c(f) \ln f \]
\[ \frac{ds}{dt} = - \int dx \int dp \int dp' \rho(p) \rho(p') \left( \ln + 1 \right) \left( F(P) \left( \mathcal{F}(p') \mathcal{F}(p')^* - \mathcal{F}(p) \mathcal{F}(p) \right) \right) \]

Lemma

\[ \int \rho(p) \rho(p') \, dp = \frac{1}{2} \int \rho(p)^* \left( \phi + \phi' \right) - \phi - \phi' \, dp \]

explicitly:

\[ \int dp \, \rho(p) \rho(p') = \int \rho(p)^* \left( \phi + \phi' \right) - \phi - \phi' \, dp \]

To show:

Now on \( \Omega \):

> interchange \( \rho, \rho' \leftrightarrow \rho', \rho \)

\[ \text{flip about} \quad \begin{cases} \text{use} \; w = w^T \quad \text{(micro-reversibility)} \end{cases} \]
\[ \int d^4 \phi \ e c_c A \]
\[ = \int d^4 \phi \left\{ (e \phi - e \phi') \ W \left( \phi, \phi', \phi', f, f' \right) \right\} \]

Now, consider:

\[ \phi \rightarrow \phi' \]
\[ \phi' \rightarrow \phi \]

and interchange above

\[ \phi, \phi' \] with \[ \phi, \phi' \]

\[ \text{n.b.: up-down symmetry equivalent} \]
\[ \text{(no reason)} \]

\[ \int d^4 \phi \ c_c A \ e = \frac{1}{2} \int d^4 \phi \left\{ (e \phi - e \phi') \right\} \]
\[ + \ e \phi (\phi) - (e \phi (\phi')) W f, f' \]

Proves Lemma 1
Now, let $\psi = \ln f$

so, from Lemma:

$$\frac{ds}{dt} = \frac{1}{2} \int dx \int d^4\phi \left( \ln f + \ln \psi - \ln f' - \ln \psi' \right)$$

$$= \frac{1}{2} \int dx \int d^4\phi \left( \psi' \ln \frac{\psi'}{\psi} \right)$$

Define $x = \frac{\psi'}{\psi}$

$$\frac{ds}{dt} = \frac{1}{2} \int dx \int d^4\phi \left( \psi' \ln x \right)$$

Since: $\int C(x) dP = 0$

have $\int w \psi \left( x - 1 \right) d^4\phi \ dx = 0$
as a case of writing zero in a complicated way!

so adding 0 to $dS/dt$ expression:

$$\frac{dS}{dt} = \frac{1}{2} \int \int dx \ w \ f(x) \left[ x \ln x - x + 1 \right]$$

no entropy production rate.

Then

$$F(x) = x \ln x - x + 1$$

$$F_1 = 1 + \ln x - 1$$

$$F(0) = 1$$

$$F(1) = 0$$

$$\frac{dS}{dt} \geq 0 \quad \rightarrow \quad H-\text{Theorem}$$
Now, \[ \frac{dS}{dt} = 0 \quad \text{for} \quad x = 1 \]

\[ f f_1 = f' f'_1 \]

\[ \ln f + \ln f_1 = \ln f' + \ln f'_1 \]

\[ \Rightarrow \ln f = c + f' V - \kappa E \]

\[ \frac{dS}{dt} = 0 \quad \Rightarrow \quad \text{Maxwellian Distribution} \]

\text{Note:}

1) \text{Keys:} \quad W = W^T \quad \Rightarrow \quad \text{Detailed Balance}

\[ f(a, z) = f(a) P(z) \quad \Rightarrow \quad \text{Factorization} \]

\[ \text{(Molecular chaos)} \]

2) \[ \frac{dS}{dt} = 0 \quad \Leftrightarrow \quad \text{CCE} = 0 \]

\text{Collisions drive system to equilibrium.}
3) $dx$ irrelevant

Entropy produced locally.

i.e. $P$ relaxes to local Maxwellian

then to uniform Maxwellian (i.e. transport: $\frac{r_0}{r(t)} \rightarrow \frac{1}{r(t)}$)

Essence of $H$ Theorem:

Macrosopic irreversibility from microscopically reversible dynamics.

$+$ Molecular chaos ("micro-chaos").
Some observations:

1. If no a-priori concept/idea of equilibrium distribution, how derive it?

Recall: \( \frac{dS}{dt} = 0 \), for \( x = 1 \)

\[ x = 1 \Rightarrow S' = S \]

\[ \ln S' + \ln S = \ln S + \ln S \]

as labels in collision arbitrary, i.e.

\[ \text{const.} \]

\[ \text{etc.} \]

So \( \ln S + \ln S = \text{const.} \) 

own laws conserved

What is conserved? (Dynamically):
- energy (kinetic energy/particle)
- momentum
- number

So, but can be expressed as a linear combination of conserved quantities

$$\ln F = a + b \cdot p + c \frac{p^2}{2m}$$

$c < 0$ for normalizability

N.B.: Angular momenta not independent as collision event at 1 position.

$$F = c' \left[ \frac{-p^3}{2amT} + \frac{p \cdot V}{T} \right]$$

$c' \sim V$ and $\frac{p \cdot V}{T}$ can be $\eta(x), T(x), V(x)$ can all be functions of $x$ for large $L$
Thus, derived equilibrium distribution function from H-Thm.

2) How reconcile?
- reversible Hamiltonian dynamics
- $dS/dt \geq 0$

Related: What happened to Poincaré Recurrence?

Point:
- statistical description: $P(x, p, t)$
- coarse graining: (recall Lyapunov exponents)

$A \Delta P$ $\rightarrow$ sets resolution scale.
(S is integrated quantity)
Why Significant?

- Partition kills small details in phase volume evolution.

$A_0 \quad t=0 \quad \rightarrow \quad \text{exact}$

With coarse-graining (smearing, in entropy production timescale)

$\frac{\rho_0 A_0}{A_{cg}} \quad \rightarrow \quad \text{coarse-grained density modified by coarse-graining}$

$F = \frac{\rho_0 A_0}{A_{cg}}$  \quad \text{coarse-grained distribution}$

$F < \rho_0 \text{ as } A_0 < A_{cg}.$
So

- prediction of close recurrences impossible as partition set resolution limit.

N.B.: $\eta_{\text{map}} = \frac{1}{\sqrt{n}}$

$\gamma_c^{-1} = \gamma_c = \sqrt{n}/\eta_{\text{map}}$

etc