Theorem. The following are equivalent (and define trees):

1. $G=(V,E)$ is connected and acyclic (no subgraph)
2. Every pair of vertices is connected by a unique path (no repeats)
3. $G$ is connected and $|E|=|V|-1$
4. $G$ is acyclic and $|E|=|V|-1$
5. $G$ is acyclic and adding any new edge creates a cycle.

$|E|=5$
$|V|=4$

Proof. Show that $1 \rightarrow 2$, $2 \rightarrow 3$, $3 \rightarrow 4$, $4 \rightarrow 1$

$5 \rightarrow 1$. $1 \rightarrow 5$

$4 \rightarrow 1$ (Frobenius)
$1 \rightarrow 2$. Use contradiction

Let $u,v$ be two vertices $\rightarrow$ not connected, then $G$ is not connected

Assume there is a path $u \rightarrow w \rightarrow v$

$1 \rightarrow 3, 4$. Use contradiction

Assume $G$ is not connected

Let $i$ be the largest number $s.t. x_i$ is in $P_i$ as $y_i$. (if no such $x_i$, use $w$)

Then $(u, x_0, x_1, \ldots, x_i, y_i, y_{i+1}, \ldots, y_n, w)$ is a cycle.

$2 \rightarrow 3$. Strong induction on $N$

Base case - $1 \rightarrow 1$, obvious

Inductive step - Assume true if $1 \rightarrow k$

If $|V|=1+k$, let $u \in V$ and $E$ be a $G=(V,E)$

In $G$, $u \rightarrow v$ must be disconnected, otherwise there would be a path from $u$ to $v$ in $G$.

Then $G$ has two connected components $G \rightarrow G_1 \rightarrow G_2$

$G_1 \rightarrow (V_1, E_1)$
$G_2 \rightarrow (V_2, E_2)$

$G_1 \cup G_2$ satisfy 2, so $|E_1|+|V_1|+1$ for $i=1,2$

$E_1=\{E, \ldots, E_{n-1}\} \cup \{x_0, x_1, \ldots, x_n \}$
$|V_1|=|V|, |V_2|=|V|-1$

$5 \rightarrow 1$. Use contradiction. Assume $G$ is not connected

In particular, $u \rightarrow v$ be two vertices not connected.

Adding $(u,v)$ to $E$ can not create a cycle. Contradicts 5.

$1 \rightarrow 5$. If we add $(u,v)$ then is a path from $u$ to $v$

$(u,v, x_0, x_1, \ldots, x_n, v)$. Adding $(u,v)$ turns this into a cycle.

$x_i$ is not in $G$. Then adding $x_i$ to $G$ must add an edge, $(x_i, x_{i+1})$ to $G$. (If $i=n$).