The Dust Subdisk in the Protoplanetary Nebula

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We present a self-consistent computation of the structure of the dust subdisk in the protoplanetary nebula. The main physical processes governing the dynamics of the dust disk are reviewed. A (nonlinear) vertical diffusion equation for the transport of dust particles is derived. It is based on a competition between sedimentation processes due to gravity and diffusion due to turbulence. The vertical structure of the subdisk is computed by solving numerically the diffusion equation. The influence of both the particle size and the strength of the turbulence is studied. Large particles are found to settle down toward their equilibrium distribution in a turbulent diffusive time scale. Small particles remain mixed throughout the whole gas disk. Simple analytical estimates of the dust scale height are given. They are found to agree closely with the exact numerical solutions. The implications of our results for cosmochemistry and the structure of the solar nebula are discussed. © 1995 Academic Press, Inc.

1. INTRODUCTION

Radiation observed from young pre-main-sequence (PMS) stars in the infrared and submillimeter wavelength ranges are usually interpreted as originating from dust-rich protoplanetary disks (e.g., Beckwith et al. 1990). For the purposes of studying planet formation it is important to understand the processes and dynamics of such disks, their evolution and final dispersal. Apart from the astronomical constraints, we also have Solar System constraints (from planets, comets, meteorites—see, e.g., Morfill and Wood 1989) and assuming that our Solar System is "representative" and not widely anomalous, a great deal of work has been done to reconcile these sets of observations. One of the most poorly studied stages of dusty disk evolution is the transition from a dispersed to a centrally (in the midplane of the disk) condensed dust distribution—generally regarded as a key issue for understanding planet formation. This is the subject of this paper.

The presence of dust particles inside the protoplanetary disk has some importance on its dynamics. The gaseous and solid phases can interact directly via collision and they can also influence indirectly the dynamics of the other component via more complicated feedback mechanisms. For example, the presence of dust particles throughout the disk increases the local opacity, thereby making the onset of convection in the gas easier. Turbulence in the gas favors the mixing of the dust particles within the whole disk; on the other hand, it also aids sedimentation toward the midplane via coagulation-induced growth. Such processes are often used in elementary computations of the structure of the solar nebula, but they have so far never been included in a self-consistent model of the two phases. The most advanced numerical study so far has been performed by Cuzzi et al. (1993), who study the diffusion of dust particles in a turbulent solar nebula via a two-phase turbulent closure model. Inclusion of dust coagulation and other sophisticated processes remains to be done. The main drawback of the numerical approach is that the two-phase model is a stiff numerical problem (Cuzzi, private communication).

Another approach is to develop analytical models which allow an easier exploration of the parameters and provide simple approximate initial conditions which can then feed into more elaborate numerical approaches.

The purpose of this paper is to build a limited self-consistent model of the dust disk in the presence of an external prescribed turbulence. Some elementary coupling between dust and turbulence will be included, but the feedback of the dust onto the turbulence (e.g., by the opacity mechanism) will be ignored. This will allow a
simple description of the dust subdisk, for a given size distribution of particles. Other models for the dust subdisk have already been proposed (Weidenschilling 1977, 1980, 1988). Our approach differs from these models by the treatment of the effect of turbulence on dust settling: in Weidenschilling's models, turbulence influences this settling directly by advecting the dust away from the midplane at a turbulent rms velocity. Such a treatment is, however, incorrect, because it does not take into account the randomness of the direction of the turbulent velocity and the fact that settling may even be accelerated by eddies with rms velocity toward the midplane. A correct approach involves averaging the advective transport by eddies in all the directions. It is well known (see, e.g., Frisch 1989) that such averaging transforms the advective effect of each individual eddy into a collective diffusive effect: the dust is therefore diffused by turbulence, rather than advected, as assumed by Weidenschilling (see also Morfill 1985, Morfill and Völk 1985).

The outline of our paper is as follows: in Section 2, we discuss the main physical processes governing the dust dynamics; in Section 3, we derive a vertical diffusion model, which includes all of these processes; the diffusion equation is solved numerically for various turbulence characteristics and grain sizes; analytical estimates of the dust subdisk scale height are provided and shown to be in good agreement with numerical solutions. Implications of our results for cosmochronological and physical processes are discussed in Section 4 and an outlook of the role of the dust feedback onto the turbulence is provided. A table of symbols used throughout this paper is provided in the Appendix.

2. PHYSICAL PROCESSES IN THE DUST DISK

2.1. The Gaseous Background

Throughout this Section, we consider various processes occurring in the dust subdisk, which is embedded in a turbulent gaseous disk. All quantities referring to the dust component are denoted with small letters or a subscript $d$, the capital letters and $g$ subscript being used for the gaseous component. For example, $h_g$ and $H$ refer, respectively, to the dust disk and the gas disk vertical scale height. For obvious symmetry consideration, we adopt a cylindrical system of coordinate $(r, \phi, z)$. The corresponding velocity components shall be denoted $(u, v, w)$ for dust particles and $(U, V, W)$ for gas particles. We shall consider the dynamics of only the dust component and neglect any back reaction on the gas component. The latter is therefore assumed to follow the usual equations of a gaseous solar nebula (see, e.g., Morfill 1985). In this paper, we shall need only two properties of the gas component, namely that it is in hydrostatic equilibrium in the vertical direction and that it is turbulent, without specifying the source of the turbulence (see, however, discussion in Section 4). By the first property, we can write the $z$ dependence of the gas density,

$$\rho_g(r, z) = \rho_{0g}(r) \exp\left[-z^2/2H^2\right],$$  \hspace{1cm} (1)

where $H$ is the gas scale height, given in terms of the sound velocity $c_s = (dP/d\rho_g)^{1/2}$, and of the orbital frequency $\Omega$ by

$$H = \frac{c_s}{\Omega}.$$  \hspace{1cm} (2)

Relation (1) holds provided that the $z$ variations of the sound velocity are neglected.

The turbulent state of the gas is completely specified provided that one knows the energy spectrum, $E(k) = \langle |V(k)|^2 \rangle/k$, where $V$ is the rms turbulent velocity, and the size of the turbulent eddies is $\lambda = (2\pi/k)$, with $k$ the corresponding wavenumber. The wavenumber and velocity of the largest eddy are $k_0$ and $V(k_0)$. The energy spectrum is assumed to follow a power law,

$$E(k) = \frac{V^2(k_0)}{k_0} \left(\frac{k}{k_0}\right)^{-\gamma},$$  \hspace{1cm} (3)

where $\gamma$ is an exponent depending on the nature of the turbulence. For example, $\gamma = 5/3$ for isotropic, incompressible turbulence, while $\gamma = 2$ for compressible turbulence. In general, $5/3 < \gamma < 3$ (see, e.g., Dubrulle and Valdettaro 1992). Because of the rotation of the solar nebula, the typical turnover time of the largest eddy is of the order of $\Omega^{-1}$ (Dubrulle and Valdettaro 1992). Therefore, $k_0 V(k_0) = \Omega$. The efficiency of the turbulence can be measured by the turbulent viscosity $\nu_t = V(k_0)/k_0$ or equivalently by a nondimensional factor, $\alpha$, which is (Shakura and Sunyaev 1973)

$$\alpha = \frac{\nu_t \Omega}{c_s^2}.$$  \hspace{1cm} (4)

In our case,

$$\alpha = \frac{1}{(k_0 H)^2}.$$  \hspace{1cm} (5)

2.2. Dust Collision and Friction

A dust particle moving at a relative velocity $\mathbf{v} - \mathbf{V}$ through the gas is decelerated due to a drag force according to
\[ \frac{dv_D}{dt} = -\frac{(v - V)}{\tau_f}, \]

where \( \tau_f \) is a characteristic friction time scale. This time scale depends on the size and the relative velocity of the particle and thus on the radial distance from the Sun. For simplicity, we shall not use this dependence explicitly in our computations and work only in terms of \( \tau_f \). We shall then define as “small particles” (respectively, “large particles”) particles with a friction time much smaller than the orbital period \( \Omega^{-1} \) (respectively, much larger). The actual size of the particles involved in this definition will depend on the radial location and on the solar nebula dynamics.

When order of magnitude estimates are needed (e.g., in the discussion), we shall follow the formulation of the friction time scale by Weidenschilling (1977). It depends on the size \( a \) of the particle relative to the mean free path \( \lambda \) of a gas molecule,

\[ \tau_f = \frac{\rho_s \lambda}{\rho_g C_D}, \quad a < \frac{9}{4} \lambda, \]

\[ \tau_f = \frac{8 \rho_s}{3 \rho_g C_D (v - V)}, \quad a > \frac{9}{4} \lambda, \]

where \( \rho_s \) is the material density, \( C_D \) is a coefficient which depends on the particle Reynolds number \( Re_p = 2a(v - V)/\nu_g \) (see Weidenschilling 1977). \( \nu_g \approx c_s \lambda \) is the kinematic viscosity of the gas. The first regime corresponds to the Epstein regime, the second to the Stokes regime. For typical solar nebula models, the Epstein regime is valid for particles smaller than 1 cm at 1 AU or 600 cm at 10 AU (Cuzzi et al. 1993). Note that it is characterized by a friction time independent of the particle relative velocity. A similar behavior can be obtained in the Stokes regime, for particles such that \( Re_p < 1 \). In that case, \( C_D = 24 Re_p^{-1} \), and the velocity dependence in \( \tau_f \) cancels. Such a regime is valid for particles between \( 1 \) and \( 10 \) cm at 1 AU, and between 600 and 1000 cm at 10 AU (Cuzzi et al. 1993). For larger particles, the friction time depends on the particle velocity. These particles are characterized by \( \Omega \tau_f > 1 \) and the corresponding friction time will not be needed precisely in our discussion.

### 2.3. Gas–Dust Dynamics

The main equations governing the gas–dust dynamics in a turbulent nebula can be written in cylindrical coordinates as follows (Morrill 1985):

For the gas component,

\[ \frac{dU}{dt} = g_z + \eta \frac{u - U}{\tau_f} - \frac{1}{\rho_g} \frac{\partial}{\partial r} \left( \rho_g \frac{V^2}{r} \right), \]

\[ \frac{dL}{dt} = \eta \frac{l - L}{\tau_f} + \text{viscous}, \]

\[ \frac{dW}{dt} = g_z + \eta \frac{w - W}{\tau_f} - \frac{1}{\rho_g} \frac{\partial}{\partial z} \left( \rho_g \frac{V^2}{r} \right), \]

where \( l = rv \) and \( L = r V \) are the specific angular momenta and \( g_z \) and \( g_r \) are the radial and vertical components of the gravitational force due to the sun.

For the gas component,

\[ \frac{dU}{dt} = g_z + \eta \frac{u - U}{\tau_f} - \frac{1}{\rho_g} \frac{\partial}{\partial r} \left( \rho_g \frac{V^2}{r} \right), \]

\[ \frac{dL}{dt} = \eta \frac{l - L}{\tau_f} + \text{viscous}, \]

\[ \frac{dW}{dt} = g_z + \eta \frac{w - W}{\tau_f} - \frac{1}{\rho_g} \frac{\partial}{\partial z} \left( \rho_g \frac{V^2}{r} \right), \]

where \( \eta = \rho_d/\rho_g \) is the abundance ratio between the dust and the gas, \( P \) is the gas pressure, and “viscous” is the term corresponding to the viscous dissipation of the gas angular momentum.

For time scales long compared with the friction time and short compared with the viscous time at which the gas angular momentum is dissipated, both the radial velocities and the specific angular momentum can be considered as stationary (\( \partial_t = 0 \)). Since both the gas and the dust are in approximate Keplerian motion, both the ratio \( u/v \) and \( U/V \phi \) are small. To first order in this parameter, we get then from (8) and (9) an estimate of the radial drift when \( \Omega \tau_f \) is small:

\[ u - U = \frac{1}{1 + \eta \frac{\tau_f}{\rho_g}} \left( \frac{\tau_f}{\rho_g} \frac{\partial}{\partial r} \left( \rho_g \frac{V^2}{r} \right) \right). \]

Equation (10) was obtained by Nakagawa et al. (1986).

As shown by Nakagawa et al. (1986), the presence of the dust particles does not affect the hydrostatic equilibrium in the vertical direction, so that we may assume that \( W = 0 \), and \( w - W = w \). The equation governing the dust vertical velocity then becomes

\[ \frac{dw}{dt} = -z \Omega^2 \frac{w}{\tau_f}, \]

where we have set the vertical gravity equal to \( -z \Omega^2 \). Using \( w = dz/dt \), one can integrate this equation to find the motion and the velocity of the dust particles in the equatorial plane (Nakagawa et al. 1986). The result depends on the particle size. Small particles, with \( \Omega \tau_f < 1 \),
are settled exponentially at a rate $\Omega \tau_f$, reaching rapidly a terminal velocity:

$$w = w - W = -z\Omega^2 \tau_f.$$  \hspace{1cm} (12)

Larger particles undergo overdamped oscillations around the midplane, with a period of the order of $\Omega^{-1}$ and a characteristic decay time of the order of $(\Omega^2 \tau_f)^{-1}$. The corresponding motion then traces a spiral in the phase plane $(z, w)$, and no functional analytical form can be obtained, relating $w$ and $z$. This sets a conceptual difficulty adapting our analytical model to the large particles case, because our diffusion equation requires the input of a functional form for the vertical drift $w - W = f(z)$. Maybe this functional form could be obtained in the future by a careful analysis of, e.g., numerical simulations of Cuzzi et al. For the time being, however, we restrict our analysis to the case of small particles ($\Omega \tau_f < 1$).

2.4. Turbulent Transport

In a turbulent medium, the frictional drag induces a stochastic velocity component $(\delta v^2)^{1/2}$ onto the dust grains (Saffman and Turner 1956). This velocity was estimated by Volč et al. (1980), under the assumption that the back reaction of the dust onto the turbulence is negligible (see Section 4 for a discussion). It is proportional to, but smaller than the mean rms velocity of the turbulent eddy,

$$\langle \delta v^2 \rangle = \int_{k_0}^{\infty} dk E(k) \frac{\tau_k \arctan(B_k)}{\tau_k + \tau_f} B_k,$$  \hspace{1cm} (13)

where $E(k)$ is the energy spectrum, given in (3), $\tau_k = 1/kV(k)$ is the turnover time of an eddy of size $2\pi/k$, and $B_k$ is a nondimensional factor, which depends on the equilibrium velocity reached by a dust particle in the turbulent wave field, $v_z$:

$$B_k = \frac{kv_z\tau_z \tau_f}{\tau_k + \tau_f}.$$  \hspace{1cm} (14)

A dust particle cannot reach the mean turbulence velocity for two reasons. The first one is the finite life time of a given eddy of size $2\pi/k$. This prevents small eddies from picking up a dust grain with characteristic friction time larger than their turnover time ($\tau_f > \tau_z$). This effect is contained in the factor $\tau_z/(\tau_z + \tau_f)$. A second efficiency reduction comes from the systematic velocity of the dust grain $v_z$. Due to this velocity, a given dust grain can cross an eddy of size $2\pi/k$ in $\tau_z = 2\pi/(kv_z)$. If this crossing time is much less than the friction time, the dust grain cannot be picked up by the eddy; if it is much larger than the eddy turnover time, the dust grain samples many eddies during one "traversal." Each of these eddies then communicates a given velocity, with random direction. On the average, this produces a reduction of the mean velocity of the dust particle. This effect is contained in the factor $\arctan(B_k)/B_k$.

Due to the random velocity (13) imparted on the grains by the turbulent gas they are transported diffusively. The corresponding turbulent diffusion coefficient can be estimated by simple dimensional arguments. A given eddy of size between $1/k$ and $1/k + dk/k^2$ communicates to a dust grain a typical random velocity,

$$\delta v(k)^2 = dk E(k) \frac{\tau_k \arctan(B_k)}{\tau_k + \tau_f} B_k,$$  \hspace{1cm} (15)

on a typical length $2\pi/k$. On dimensional grounds, the corresponding turbulent diffusivity $d\kappa_k$ can be estimated as

$$d\kappa_k = \frac{\delta v(k)^2}{k^2}.$$  \hspace{1cm} (16)

The mean diffusivity induced by all the turbulent eddies is

$$\kappa_t = \left( \int_{k_0}^{\infty} d\kappa_k \right)^{1/2}.$$  \hspace{1cm} (17)

To complete the definition of $\kappa_t$, it remains to estimate the velocity $v_z$ appearing in $B_k$ (Eq. (14)). This velocity should normally be estimated in a self-consistent way, via

$$v_z^2 = v_z^2 + \langle (V(k) - \delta v)^2 \rangle,$$  \hspace{1cm} (18)

where $v_z$ is the systematic velocity due to drift plus gravity. The drift velocities in the radial and vertical direction were computed in Section 2.3, Eqs. (10) and (12). The ratio of the radial drift to the vertical drift is of the order of

$$\frac{u - U}{w - W} \sim \frac{H H}{z r}.$$  \hspace{1cm} (19)

The radial drift is therefore important only near the midplane, for $z/H < (H/r)$. At sufficiently large $z$ it can be neglected with respect to the vertical drift:

$$v_z^2 = z^2 \Omega^4 \tau_z^2.$$  \hspace{1cm} (20)

The second term in the right-hand side of (18) is at most of the order of $k_0 E(k_0) \sim \Omega^2/k_0^2 \sim \alpha^2 c_f^2$. It is probably
FIG. 1. The normalized turbulent diffusivity with a spectral index \( \gamma = 5/3 \) dependent on the characteristic height \( k_0z \) for different \( \Omega \tau_f \). Solid lines, exact integration of the whole spectrum after Eq. (21). Dotted lines, approximation where only the largest eddies contribute to turbulent diffusivity (Eq. (23)).

smaller than that, because for very large or very small eddies one of the two reduction factors in Eq. (15) becomes irrelevant. We shall therefore neglect it compared to the systematic drift.

Thus we finally obtain the turbulent diffusion coefficient for dust particles

\[
\kappa_i = \left( \int_0^\infty \frac{dk}{k^2} \frac{k^2}{\tau_k + \tau_f} \right)^{1/2} F(k_{0z}, B_k),
\]

with \( B_k \) given by

\[
B_k = k_0z(\Omega \tau_f)^2 \frac{\tau_k}{\tau_f + \tau_k}.
\]

From formulae (21) and (22), it is apparent that the turbulent diffusivity is position dependent. We recall that the main origin of this vertical dependence is the vertical settling of the dust particles which causes crossing of eddies (see discussion after Eq. (14)). The largest value for \( \kappa_i \) is reached at the midplane, where \( B_k = 0 \). It then decreases steadily toward the disk surface, like \( z^{-1/2} \). The turbulent diffusivity has been computed numerically for several turbulence spectra, several particle sizes, and at various positions. The results are given in Fig. 1. The value of the power spectral exponent, \( \gamma \), has little influence on the final outcome. As expected, small particles, for which \( \Omega \tau_f \) is small, experience an almost constant turbulent diffusivity. For larger values of \( \Omega \tau_f \), however, the diffusivity drops noticeably between the midplane and the disk surface.

The turbulent diffusivity is computed by integration over the whole spectrum. In fact, most of the power resides at wavenumbers close to \( k_0 \). Therefore, the contribution to \( \kappa_i \) is dominated by its value at \( k_0 \) and the turbulent diffusivity (21) can be approximated by

\[
\kappa_i = \kappa_{i0} \left( \frac{\int_0^\infty dk E(k)}{k^2} \right)^{1/2}
\]

\[
= \frac{\Omega}{\sqrt{1 + \gamma k_0^2}} \kappa_{i0},
\]

where \( \kappa_{i0} \) is given by

\[
\kappa_{i0} = \frac{1}{\Omega \tau_f} \left( \frac{\arctg(Fk_{0z})}{k_{0z}} \right)^{1/2}.
\]

\( F \) is a nondimensional form factor:

\[
F = \frac{(\Omega \tau_f)^2}{1 + \Omega \tau_f}.
\]

Note that since \( \kappa_{i0} \propto k_{i0} \), it can also be viewed as the inverse Schmidt number (Cuzzi et al. 1993). Figure 1 also shows the comparison between the exact formula (21) and the approximation (23), given as dotted lines. The approximation yields a slightly smaller turbulent diffusivity, because contributions from the whole turbulence cascade are neglected, but the numerical agreement is nevertheless satisfactory. From now on, we shall therefore work only with the approximate formula (23). There are two interesting limit cases for \( \kappa_i \):

\[
\kappa_i = \frac{1}{\sqrt{\gamma + 1}} \frac{\Omega}{k_0^2}, \quad \Omega \tau_f \ll 1,
\]

\[
\kappa_i = \left( \frac{1}{2(\gamma + 1)k_{0z}} \right)^{1/2} \frac{\Omega}{k_0^2}, \quad \Omega \tau_f \gg 1.
\]

The first limit is not too surprising: in the limit \( \tau_f \to 0 \), the dust particles become indistinguishable from gas particles, and their diffusivity tends to the gas turbulent diffusivity. The second limit merely states that the larger the particle (or boulder) the less effective the gas drag induced random motion. Of course, our treatment here is not appropriate for large particles. The effects of stochastic gas drag on "boulders" have to be described in terms of the integrals of motion and the associated cyclic variables (see, e.g., Hassan and Wallis 1983). Nonetheless, these analytical limits serve as good tests of the numerical procedures. Our formulation for the turbulent diffusivity can be, for example, compared to that adopted
by Cuzzi et al. (1993), which translates in our notations into
\[ \kappa_{\text{Cuzzi}} = \frac{\Omega}{k_0^2} \frac{1}{1 + \Omega \tau_f \sqrt{1 + (k_0 \zeta)^2 (\Omega \tau_f)^2}}. \]

These two formulations are identical in the limit \( \Omega \tau_f \ll 1 \). They differ in limit \( \Omega \tau_f \gg 1 \) by the exponent of the factors \( k_0 \zeta \) and \( \Omega \tau_f \), but show the same overall dependence.

3. VERTICAL DIFFUSION

3.1. The Model

The physical processes reviewed in Section 2 are valid for dust of a given size \( a \). In a real mixture, the dust component is characterized by a whole size distribution \( f(a) \). This size distribution evolves via collisions and coagulation. Here we shall ignore this complication and consider only the fate of a given size distribution, which is assumed to be sharply peaked around a given characteristic size \( \langle a \rangle \). This simplification will be discussed in Section 4; see also Morfill (1985).

The dust component placed in a turbulent gaseous field is transported diffusely. The transport equation is, in the absence of source and sink terms,
\[ \partial_t \rho_d + \nabla \cdot (\rho_d (\mathbf{v}_d - \mathbf{V})) = \nabla \cdot \left( \rho_d \kappa \nabla \left( \frac{\rho_d}{\rho_g} \right) \right), \]

where \( \mathbf{v} - \mathbf{V} \) is the mean relative velocity of the dust particles with respect to the gas and \( \kappa \) is the turbulent diffusivity given by
\[ \kappa = \kappa_0 \left( \int_{k_0}^{\infty} dk E(k) k^2 \right)^{1/2} \]
\[ = \frac{\Omega}{\sqrt{1 + \gamma k_0^2}} \kappa_0, \]

where \( \kappa_0 \) is given by
\[ \kappa_0 = \frac{1}{\Omega \tau_f} \left( \frac{\arctg[F k_0 \zeta]}{k_0 \zeta} \right)^{1/2}. \]

\( F \) is a nondimensional form factor:
\[ F = \frac{(\Omega \tau_f)^2}{1 + \Omega \tau_f}. \]

Since the disk is thin, vertical gradients are much larger than radial gradients. The vertical structure is therefore established much faster than the radial evolution, so we may ignore the latter for our purposes:
\[ \partial_t \rho_d + \partial_z (z \Omega^2 \tau_f \rho_d) = \partial_z \left( \rho_d \kappa \partial_z \left( \frac{\rho_d}{\rho_g} \right) \right). \]

We have used Eq. (12) for the vertical relative velocity. \( \kappa \) is the turbulent diffusivity given by (30). This equation can be written in a simple nondimensional form by using \( H \) and \( 1/(\Omega^2 \tau_f) \) as characteristic length and time scale and by introducing the dust to gas ratio \( \eta = \rho_d/\rho_g \). Note that both \( H \) and \( \Omega \tau_f \) are only \( r \) dependent. In nondimensional form, the diffusion equation is
\[ \partial_t \eta + (\xi^2 - 1) \eta - \xi (1 - \kappa) \partial_z \eta = \partial_z (\kappa \partial_z \eta), \]
with \( \tau = t \Omega^2 \tau_f, \eta = z/H, \) and \( \kappa = \kappa/(H^2 \Omega^2 \tau_f) \). We have used the isothermal expression for the gas density (Eq. (11)). Equation (33) is a nonlinear diffusion equation with nonconstant diffusion coefficient. Its solution has therefore to be sought numerically. Some simple analytical estimates can, however, be given.

3.2. Analytical Estimates

Equation (33) admits a simple family of stationary solution, of the form
\[ \eta = \eta_0 \exp[-\omega \xi^2/2], \]
\( \omega \) being given by
\[ \omega^{-1} = \kappa \tau_f. \]

From Eq. (34) we see that the scale height of \( \eta \) is given by
\[ \frac{h}{H} = \frac{1}{\sqrt{\omega}}. \]

The scale height found with formula (36) is
\[ \frac{h}{H} = \left( \frac{1}{\gamma + 1} \right)^{1/2} \sqrt{\frac{\alpha}{\Omega \tau_f}}. \]

Note that our estimate differs from Weidenschilling's estimate, which leads to
\[ \frac{h}{H} = \frac{\alpha^{1/2}}{\Omega \tau_f}. \]

This discrepancy arises essentially because Weidenschilling adopted a fixed size scale \(( \propto H) \) rather than a fixed...
turnover time as we did (see discussion in Section 2.1 about this point).

The dust subdisk scale height \( h_d \) can be related to \( h/H \) using the definition of \( \eta \). We find

\[
\frac{h_d}{H} = \frac{h}{H} \left[ 1 + (h/H)^2 \right]^{-1/2}.
\]  

(39)

Therefore, for small values of \( h/H, \) the dust scale height is essentially equal to \( h \). As \( h/H \) increases, it becomes close to the gas scale height. Using Eq. (39), it can be seen that our treatment predicts thinner dust subdisks than Weidenschilling's treatment. We shall come back to this point in the discussion.

The family of solutions (34) can be used as an estimate for the real solution for any particle size. However, for very small particles, \( \Omega \tau_f \ll 1 \), an exact stationary solution can be found. In that case, \( \kappa = \kappa_c = \alpha (1 + \gamma)^{-1/2} (\Omega \tau_f)^{-1} \) is independent of \( \xi \) and we get

\[
\eta = \frac{\eta_0 e^{-\xi^2/(\Omega \tau_f)}}{1 + \eta_0 (1 - e^{-\xi^2/(\Omega \tau_f)})}.
\]

(40)

This stationary solution is reached in a nondimensional time scale \( \tau = 1 \), i.e., \( \tau_{\text{stat}}^{-1} = \Omega^2 \tau_f \).

### 3.3. Numerical Solution

We discretized Eq. (33) using a central difference scheme in order to calculate the time-dependent evolution of an initially homogeneously distributed gas/dust mixture with standard cosmic abundance \( f_c = 0.01 \). The dust to gas ratio is expected to approach zero at a few scale heights above the midplane. A reasonable guess for the expected scale height can be obtained from the analytical estimate (see Eq. (37)). These values are taken as inputs for boundary values of the dust to gas ratio defined at four scale heights. We notice that the stationary solution is only weakly dependent on the actual boundary position, if it exceeds several scale heights. As an example we show in Fig. 2 a typical evolution calculation for \( \alpha = 0.002 \) and \( \Omega \tau_f = 0.1 \). The stationary solution adjusts within a few time scales \( \tau_{\text{stat}} \). A characteristic half-width of \( (h/H)_{1/2} = 0.104 \) can be extracted. The total amount of dust material remains constant with time during the evolution and enables to normalize the function \( \eta \):

\[
\Sigma_d = \int_{-\infty}^{\infty} dz \, \eta(z, t) \cdot \rho_g = f_c \cdot \Sigma_g
\]

(41)

The stationary solutions of the vertical disk structure for four different \( \Omega \tau_f \) and \( \alpha = 0.002 \) is displayed in Fig. 3.

A summary of calculated disk half-thicknesses for a range of \( \Omega \tau_f \) and \( \alpha \) values are shown in Fig. 4. The values extracted from the nonlinear solution of the evolution equation are indicated as asterisks. The lines indicate the approximate solutions of Eq. (37). The agreement is quite good and we conclude that the approximate treatment with simple analytical estimates given above is valid over a large range of interesting parameters.

### 4. DISCUSSION

We have developed a self-consistent model of the dynamics of small dust particles in the protoplanetary disk in the presence of a prescribed gas turbulence, characterized by a typical time scale \( \Omega^{-1} \) and its \( \alpha \) parameter (Eq.
Our model was designed to describe the settling of dust within the whole solar nebula. It could potentially be used in other situations, for example when the turbulence is localized in a small portion of the disk, around the interface with the dust subdisk (as in Cuzzi et al. 1993). In that case, \( \alpha \) becomes \( h \)-dependent. Indeed, since the turbulence is due to the shear at the interface between the dust and the gas, its typical characteristic length scale is \( h \), while its typical characteristic velocity scale is of the order of the orbital velocity difference between the dust and the turbulence. In the midplane, the friction between the gas and the dust is negligible. The dust then follows Keplerian orbits, and the orbital velocity difference between the gas and the dust is mainly due to pressure gradients. We then have \( \Delta u \sim (1/\Omega r) \partial P/\partial r \). The turbulent viscosity is then of the order of \( \nu_t = C \Omega H \), where \( C \) is a constant to be computed for small particles and the nondimensional viscosity is \( \alpha = Ch/r \). Because \( \alpha \) depends on \( h \), the diffusion equation becomes an implicit equation in \( h/H \). However, if we ignore the \( h/H \) dependence in the diffusion coefficient and still use the result (37), as in the case \( \alpha = \text{cte} \), we find that the scale height follows:

\[
\frac{h}{H} = \frac{C}{\sqrt{1 + \frac{\gamma r}{\Omega \tau_f}}}.
\]

If we take \( C = 10^{-3} \), a good agreement with the values obtained by Cuzzi et al. is obtained (Valageas 1994).

Our model indicates that sedimentation of grains inside a turbulent solar nebula occurs on a time scale of the order of \( (\Omega^2 \tau_f)^{-1} \), producing a dust layer with relative thickness given by \( h/H \approx \frac{1}{2} \), where \( \kappa_r = \kappa_r / (H^2 \Omega^2 \tau_f) \) and \( \kappa_r \) is given by (26). This assumes no coagulation, or slow coagulative grain growth at a rate \( \lesssim \Omega^2 \tau_f \). The characteristic settling time to form a stationary subdisk, \( \tau_{\text{sett}} \), is exactly the settling time scale in a nonturbulent gas. For a solar nebula with \( \Sigma_g = 100 \, \text{g/cm}^2 \), this time scale is of the order of a few Myr at 2.77 AU for micrometer-sized particles. Therefore, turbulence does not influence the time scale of sedimentation; it merely determines the equilibrium scale height of the dust subdisk. From Eq. (39), it can be seen that a criterion for grain sedimentation is

\[
\frac{h}{H} < 1.
\]

Otherwise, grains are mixed throughout the disk. From Eq. (37), we see that the smallest grains, which can experience sedimentation in the presence of a turbulence of efficiency \( \alpha \) (see Eq. (4)), have a size of the order of \( a = 150 \alpha \text{ cm} \), which we call the "settling limit." For example, if the turbulence is generated by differential rotation, \( \alpha = 2 \times 10^{-3} \) (Dubrulle 1993), the smallest grains likely to sediment have sizes of about 3 mm. This estimate may be compared with the value obtained following Weidenschilling's approach, which leads (via Eq. (38)) to \( a = 100 \alpha^{1/2} \text{ cm} \) for the smallest grains likely to sediment, i.e., \( a = 4.5 \text{ cm} \) for \( \alpha = 2 \times 10^{-3} \). This finding led Weidenschilling (1980) to question the proposal by Goldreich and Ward (1973) that planetesimal formation should occur mainly via dust settling toward the midplane and gravitational instability inside the resulting dust subdisk. He pointed out that if no coagulation occurs, the primordial dust consisting of typically micrometer-sized particles, cannot settle down efficiently in a turbulent solar nebula. Our result gives a somewhat lower settling limit, but does not essentially alter Weidenschilling's argument.

Therefore, the explanation of planetesimal growth requires the existence of coagulation and/or prolonged quiescent (nonturbulent) periods. As already noted, coagulation induces an evolution of the particle size distribution \( f(a) \) and makes the analysis of the (differential) sedimentation much more complicated. However, if the time scale for coagulation is large with respect to the sedimentation time scale, the particle size distribution may be considered as time independent, and our treatment could again be applied using the mean particle size \( \langle a \rangle = \int a f(a) \, da \). In this case a decoupled treatment of coagulation, radial transport and sedimentation is allowed. Under usual solar nebula conditions, however, coagulation time scales are well in the range of sedimentation times. We can estimate the coagulation rates in a turbulent environment.
using the analytical approach for grain growth introduced by Morfill and Völk (1985). This method made possible an analytical treatment for combined coagulation and radial transport calculations that assumed continuous feeding of small interstellar grains at the outer boundary of the nebula (see also Morfill 1985, 1988). These calculations assumed no sedimentation and yielded particle sizes of less than 1–10 cm throughout most of the disk (cf. the discussion of Fig. 5).

In turn, we will now give a preliminary discussion of coagulation and settling in a turbulent disk. A full treatment is in preparation; it goes beyond the scope of this paper, however. From Morfill (1985) we get the coagulation time scale in the case of small particles in a turbulent gas,

$$\tau_{\text{coag}}^{-1} = \beta \frac{\rho_d}{\rho_s} (\Delta v_{1,2})^{1/2} \gamma_{f},$$

(44)

where the relative velocity between grains of "near equal" size ($a_1 \approx a_2$ within a factor 10) is given by

$$\langle \Delta v_{1,2}^2 \rangle^{1/2} = \langle \delta V^2 \rangle^{1/2} \frac{2 \Omega_{f}}{1 + \Omega_{f}},$$

(45)

Here $\beta$ is the sticking efficiency, $\rho_d$ the material density of the grains, and from Dubrulle and Valdettaro (1992) we get $\langle \delta V^2 \rangle = \alpha c_{s}^2$ for the mean square turbulent gas speed. The scaling factor $\gamma_{f} \sim O(1)$ depends on the narrowness of the particle size distribution about the mean radius ($a$). According to (33), the particle distribution settles to an equilibrium in a time scale

$$\tau_{\text{stat}} = \frac{1}{\Omega_{f}^2 \tau_{f}},$$

(46)

Comparing these time scales for small particles ($\Omega_{f} \ll 1$) we get

$$\frac{\tau_{\text{stat}}}{\tau_{\text{coag}}} = \sqrt{2 \alpha \beta \gamma_{f} (\Omega_{f})^{-3/2}} \eta.$$  

(47)

Substituting the drag law for small particles together with the definition $\Sigma_g = 2 \rho_s H$ we get

$$\frac{\tau_{\text{stat}}}{\tau_{\text{coag}}} = \sqrt{\frac{\alpha}{2 \beta \gamma_{f}}} \left( \frac{\Sigma_g}{\rho_s a} \right)^{3/2} \eta.$$  

(48)

A dust subdisk may form (essentially unaffected by coagulation) when $\tau_{\text{stat}} < \tau_{\text{coag}}$, i.e., for sizes larger than some critical value $a_c$:

$$a > a_c = \left( \frac{\sqrt{\alpha}}{2 \beta \gamma_{f}} \right) \frac{\Sigma_g}{\rho_s} \eta^{2/3}.$$  

(49)

For usual solar nebula parameters ($\Sigma_g = 100$ g/cm$^2$, $\rho_s = 1$ g/cm$^3$), for $\beta = 1$, $\gamma_{f} = 1$ the critical size scales as $63(\eta^2 \alpha)^{1/3}$ cm. In principle, subdisk formation starts if $\eta$ increases above the cosmic abundance $f_c$. Putting $\eta = f_c = 10^{-2}$ and $\alpha = 2 \times 10^{-3}$ gives a critical size of 3.6 mm.

In Fig. 5 we have compared the computed mean grain size for a "radial transport plus coagulation" model with this value of $a_c$. The calculation is identical to that of Morfill and Völk (1985), assuming a column density of $\Sigma_g = 100$ g/cm$^2$ throughout the disk, except that the Dubrulle and Valdettaro (1992) turbulence description was used with a value for $\alpha = 2 \times 10^{-3}$. We see that except for the outer portion of the disk, where particle sizes are small and their number densities correspondingly large, the particle sizes present in the disk are larger than the critical value $a > a_c$ or, correspondingly, $\tau_{\text{stat}} < \tau_{\text{coag}}$. On the other hand, the particles are everywhere smaller than $\tau_{f}$, a regime where our assumptions about particle vertical velocities are true. This would imply that our subdisk calculations are valid in huge parts of the disk in this particular nebula model—coagulation proceeds simply too slowly to affect the initial settling time. Further evolution occurs then on a time scale $\sim \tau_{\text{coag}}$—the particles grow when they have already settled toward the midplane, provided that radial transport and loss into the (proto) Sun is not more rapid.
However, one more caveat in interpreting the above result has to be kept in mind: Implicit in the derivation of the coagulation time-scale used here and in Morfill (1985) is the assumption that the dust particle density remains constant in time; i.e., dust density enhancement due to settling toward the midplane is small. However, as is apparent from comparing Figs. 4 and 5, settling to a thinner dust subdisk and the corresponding dust density enhancement are not negligible processes in most parts of the disk. This will have a number of consequences:

1. The turbulence in the dust disk is decreased, as has been discussed in this paper, by the decoupling from turbulent eddies with scales larger than \( h \) (leading to a residence time less than the coherence time of the eddy—the effect associated with \( \nu_\lambda \)).

2. The coagulation time scale will be modified by the corresponding change in density and collision velocity.

This problem has to be treated self-consistently. It implies a description of a highly nonlinear situation, which in principle may evolve on a (quasi) stable branch, as discussed above, or end up unstable. This will be discussed elsewhere.

**APPENDIX: LIST OF SYMBOLS**

\[(r, \phi, z)\], Cylindrical coordinates
\[v = (u, v_\phi, w)\], Dust velocity components
\[V = (U, V_\phi, W)\], Gas velocity components
\[l\] Dust-specific angular momentum
\[L\] Gas-specific angular momentum
\[\rho_d\] Gas density
\[\rho_g\] Dust density
\[\rho_m\] Material density
\[\eta = \sigma_d/\rho_g\] Abundance ratio
\[H\] Gas disk scale height
\[h\] Abundance ratio scale height
\[h_d\] Dust disk scale height
\[\xi = z/H\] Normalized vertical position
\[c_s\] Sound velocity
\[\Omega\] Gas orbital frequency
\[P\] Gas pressure
\[\gamma\] Turbulent spectral index
\[V(k)\] Velocity of gas eddy of size \(1/k\)
\[k\] Wavenumber
\[\tau_k\] Turnover time of turbulent eddy of size \(1/k\)
\[\nu_f\] Gas–dust friction time
\[\nu_t\] Turbulent viscosity
\[\kappa_t\] Turbulent diffusivity
\[a\] Dust particle size
\[\delta\] Stochastic dust velocity
\[v_t\] Dust velocity in a turbulent field
\[v_f\] Dust velocity due to drift plus gravity

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**REFERENCES**


