Math 55a Homework 4

Due Wednesday September 30, 2020.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Questions marked * may be on the harder side.

Material covered: Linear maps and matrices; quotient, dual, transpose; linear operators, invariant subspaces, eigenvectors. (Artin §4.1-4.4 / Axler chapters 3 and 5).

0. Sometime over the weekend of September 26-27, please complete the week 4 survey (in Canvas).

1. Given a field k, consider the linear operator $\phi : k^2 \to k^2$ given by $\phi(x,y) = (-y, x)$.

(a) if $k = \mathbb{R}$, show that ϕ has no nontrivial invariant subspaces (and in particular, no eigenvectors).

(b) if $k = \mathbb{C}$, find the eigenvectors and eigenvalues of ϕ.

(c) what if $k = \mathbb{F}_2$?

2. Axler exercise 5.C.16.

3. Let ϕ and $\psi : V \to V$ be linear operators on a vector space V of dimension n. Show that

$$\operatorname{rank}(\phi \circ \psi) \geq \operatorname{rank}(\phi) + \operatorname{rank}(\psi) - n.$$

4. Let V be a finite-dimensional vector space over a field k, and let $\phi : V \to V$ be a linear operator.

(a) Show that there exists a nonzero polynomial $p \in k[x]$ such that $p(\phi) = 0$. (Hint: are the elements $1, \phi, \phi^2, \dots \in \operatorname{Hom}(V,V)$ linearly independent?)

(b) Show that, if ϕ is invertible, then there exists a polynomial $q \in k[x]$ such that $\phi^{-1} = q(\phi)$.

5. Let V be a finite-dimensional vector space over a field k, and let ϕ and $\psi : V \to V$ be two linear operators. If $\phi \circ \psi = 0$, does it follow that $\psi \circ \phi = 0$? If $\phi \circ \psi$ is nilpotent (i.e., there exists m such that $(\phi \circ \psi)^m = 0$), does it follow that $\psi \circ \phi$ is nilpotent?

6. Let V be a finite-dimensional vector space over a field k. A linear operator $\phi : V \to V$ is said to be *diagonalizable* if there exists a basis for V such that the matrix representing ϕ is diagonal (equivalently, if there is a basis of V consisting of eigenvectors of ϕ). Denote by $\lambda_1, \dots, \lambda_m \in k$ the distinct eigenvalues of a diagonalizable operator ϕ. Consider the linear operator

$$\pi_i = \frac{1}{\prod_{j \neq i}(\lambda_i - \lambda_j)} \prod_{j \neq i} (\phi - \lambda_j).$$

What is the kernel of π_i? What is its image? What is the operator $\pi_1 + \dots + \pi_m$?

7*. Let V be a finite-dimensional vector space over a field k. We say that two operators $\phi, \psi : V \to V$ are *simultaneously diagonalizable* if there exists a single basis for V such that the matrices representing ϕ and ψ in that basis are both diagonal.

Show that, if ϕ and ψ are diagonalizable, then they are simultaneously diagonalizable if and only if they commute, i.e. $\phi \circ \psi = \psi \circ \phi$.

(Hint: the hard part of the argument is showing that, if ϕ is diagonalizable and a subspace $W \subset V$ is invariant under ϕ, then the restriction $\phi_{|W}$ is diagonalizable. One way to go about this is to use the result of the preceding problem.)

8. Let $\mathbb{F}_p = \mathbb{Z}/p$ denote the field with p elements, and let $V_d \subset \mathbb{F}_p[x]$ be the space of polynomials of degree at most d with coefficients in $\mathbb{F}_p$. Consider the evaluation map

$$\begin{aligned} \phi_d : V_d &\to (\mathbb{F}_p)^p \\ f &\mapsto (f(0), f(1), \dots, f(p-1)). \end{aligned}$$

(a) Show that ϕ_d is surjective for all $d \geq p-1$.

(b) By comparing dimensions, show that ϕ_p has a one-dimensional kernel.

(c) Find explicitly a generator of $\mathrm{Ker}(\phi_p)$; that is, a nonzero polynomial $f_0 \in \mathbb{F}_p[x]$ of degree at most p whose values are identically zero.

(d) By comparing dimensions, show that in general a polynomial $f \in \mathbb{F}_p[x]$ has all its values zero if and only if it is divisible by f_0.

9. Given a vector space V over a field k, a linear operator $p : V \to V$ is said to be a *projection* if $p^2 = p$.

(a) Show that if p is a projection, then $V = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$. How does p act on each summand?

(b) Assume $\mathrm{char}(k) \neq 2$. Show that, if p_1 and $p_2 : V \to V$ are projections, then $p_1 + p_2$ is a projection if and only if $p_1 \circ p_2 = p_2 \circ p_1 = 0$. What goes wrong if $\mathrm{char}(k) = 2$?

10. A *complex* $C^\bullet$ of vector spaces is a sequence

$$\cdots \longrightarrow V_{i-1} \xrightarrow{\phi_{i-1}} V_i \xrightarrow{\phi_i} V_{i+1} \xrightarrow{\phi_{i+1}} V_{i+2} \xrightarrow{\phi_{i+2}} \cdots$$

of vector spaces and linear maps such that $\phi_m \circ \phi_{m-1} = 0$ for all m; that is, $\mathrm{Im}(\phi_{m-1}) \subset \mathrm{Ker}(\phi_m)$ for all m. In this case, the m-th *cohomology group* $H^m(C^\bullet)$ of the complex is defined to be the quotient

$$H^m(C^\bullet) = \mathrm{Ker}(\phi_m)/\mathrm{Im}(\phi_{m-1}).$$

Assuming that all the vector spaces V_i are finite-dimensional, and only finitely many of them are non-zero, show that

$$\sum_m (-1)^m \dim V_m = \sum_m (-1)^m \dim H^m(C^\bullet).$$

11. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?