# EECS 336: Lecture 3: Introduction to Algorithms

### Dynamic Programming Weighted Interval Scheduling

**Reading:** 6.0-6.3

**Announcements:**

- homework due Wednesday midnight

**Last Time:**

- philosophy
- computational tractability
- runtime analysis & big-oh

**Today:**

- Dynamic Programming (a derivation)
- Weighted interval scheduling

## Dynamic Programming

"divide problem into small number of subproblems and memoize solution to avoid redundant computation"

### Example: Weighted Interval Scheduling

input:

- $n$ jobs $J = \{1, ..., n\}$
- $s_i$ = start time of job $i$
- $f_i$ = finish time of job $i$
- $v_i$ = value of job $i$

**compatibility constraint:** Only one job can run at once.

**output:** Schedule $S \subseteq J$ if compatible jobs with maximum total value.

### Find a First Decision

"make progress towards a solution"

**Idea:** job $i$ is either in OPT($J$) or not.

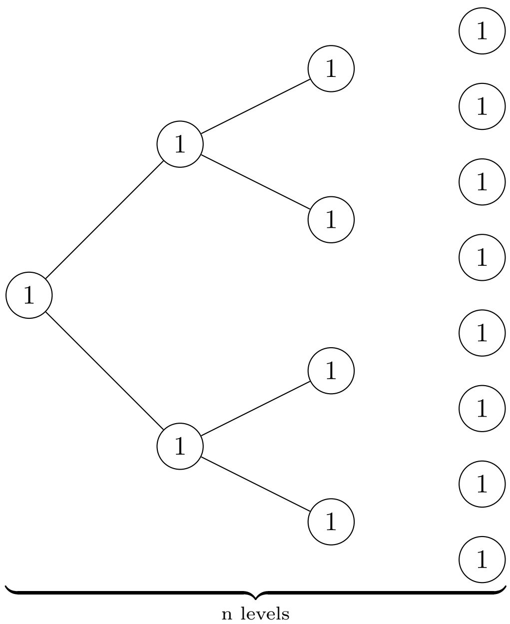
1. let $J'$ = jobs compatible with $i$ in $J$.
2. let $V$ = value of OPT if "$i \notin \text{OPT}(J)$"

   $= \text{OPT}(J \setminus \{i\})$.
3. let $V'$ = value of OPT if "$i \in \text{OPT}(J)$."

   $= v_i + \text{OPT}(J')$
4. return $\text{OPT}(J) = \max(V, V')$.

**Note:** subproblems: schedule $J'$ and $J \setminus \{i\}$.

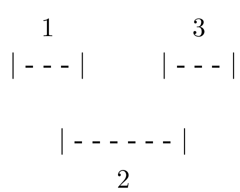
**Recurrence:** $T(n) = 2T(n-1) + 1$

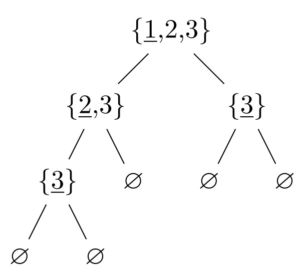



$T(n) = O(2^n)$

**Challenge 1:** redundant computation

**Example:**




**Note:** OPT({3}) called twice!

**Solution:** memoize.

"when computing the value of a subproblem save the answer to avoid computing it again"

**Result:** runtime = # of subproblems × cost to combine.

**Challenge 2:** could have too many subproblems. (could be exponential!)

**Solution:** require "succinct description" of subproblems.

**Idea:** for interval scheduling, process jobs in order of start time so subproblems suffixes of order.

- sort jobs by increasing start time, $s_1 \leq s_2 \leq ... \leq s_n$.
- let next[$i$] denote job with earliest start time after $i$ finishes. (if none, set next[$i$] $= n + 1$.)
- subproblems when processing job 1:
  - $J' = \{\text{next}[i], \text{next}[i] + 1, ..., n\}$
  - $J \setminus \{i\} = \{2, 3, ..., n\}$
- suffix $\{j, ..., n\}$ is succinctly described by "$j$".

  (only $n$ subproblems)

### Recursive Memoized Algorithm

**Algorithm:** Weighted Interval Scheduling:

1. sort jobs by increasing start time.
2. initialize array next[$i$].
3. initialize OPT[$i$] $= \varnothing$ for all $i$.
4. initialize OPT[$n + 1$] $= 0$.
5. compute OPT(1).

**Subroutine: OPT(i)**

1. if OPT[$i$] $\neq \varnothing$, return OPT[$i$].
2. OPT[$i$] $\leftarrow \max(v_i + \text{OPT}[\text{next}[i]],\ \text{OPT}[i + 1])$.
3. return OPT[$i$].

### Correctness

"OPT($i$)" is correct (by induction on $i$)

**Runtime Analysis**

- $n$ subproblems
- constant time to combine
- initialization: sorting & precomputing 'next' array

**Runtime:** $O(n)+$ initialization $= O(n \log n)$

**Iterative DPs**

"fill in memoization table from bottom to top"

**Algorithm:** iterative weighted interval scheduling

1. $\text{OPT}[n+1] = 0$
2. for $i = n$ down to 1:

   $\text{OPT}[i] = \max(v_i + \text{OPT}[\text{next}[i]], \text{OPT}[i+1])$.

**Finding Optimal Schedule**

"traverse memoization table to find schedule"

**Algorithm:** schedule

1. $i = 1$
2. while $i < n$:

   if $\text{OPT}[i+1] < v_i + \text{OPT}[\text{next}[i]]$:

   (a) schedule $i$.
   (b) $i \leftarrow \text{next}(i)$.

   else: $i \leftarrow i + 1$.

**Key Ideas of Dynamic Programming**

Subproblems must be:

1. succinct (only a polynomial number of them)
2. efficiently combinable.
3. depend on "smaller" subproblems (avoid infinite loops), e.g.,
   - process elements "once and for all"
   - "measure of progress/size."

**Seven Part Approach**

I. identify subproblem in English

   $\text{OPT}(i)$ = "optimal schedule of $\{i, ..., n\}$ (sorted by starting time)"

II. specify subproblem recurrence (argue correctness)

   $\text{OPT}(i) = \max(\text{OPT}(i+1), v_i + \text{OPT}(\text{next}[i]))$

III. solve the original problem from subproblems

   Optimal Interval Schedule = $\text{OPT}(1)$

IV. identify base case

   $\text{OPT}(n+1) = 0$

V. write iterative DP.

VI. runtime analysis.

   $O(n)$ + initialization $= O(n \log n)$

VII. implement in your favorite language (Python!)