What can linear algebra say about graphs?

Def. Adjacency matrix A of a graph G(V,E) is a matrix A \in \mathbb{R}^{n \times n}.

\[ (A)_{ij} = \begin{cases} 
1 & \text{if } (v_i, v_j) \in E \\
0 & \text{otherwise}
\end{cases} \]

Dil. Let \( \lambda \) be an eigenvalue of \( A \).

Let \[ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \] be a corresponding eigenvector.

Then \[ A x = \lambda x \]

Dil. All eigenvalues of a tree are real and negative.

Dil. Let \( T \) be a tree with \( n \) vertices.

\[ A_T = \begin{bmatrix} 
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 
\end{bmatrix} 
\]

\[ A_T = \begin{bmatrix} 
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix} 
\]

Any tree has \( n-1 \) negative eigenvalues and one zero eigenvalue.

Def. A matrix \( A \) is symmetric if \( A_{ij} = A_{ji} \) for all \( i, j \).

Dil. Let \( A \) be a symmetric matrix.

\[ A = \begin{bmatrix} 
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn} 
\end{bmatrix} 
\]

Dil. Let \( G(V,E) \) be a regular graph.

Then, \( (1, 1, \ldots, 1) \) is an eigenvector of \( A \) with eigenvalue \( n \), where \( n \) is the degree of each vertex.

Dil. Let \( A \) be an adjacency matrix, \( (x_1, x_2, \ldots, x_n) \)

Then \( (x_1, x_2, \ldots, x_n) \) is a normalized eigenvector of \( A \) with eigenvalue \( \lambda \).

Diagram:

- Graph G(V,E)
- Adjacency matrix A
- Eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \)

Conclusion:

If \( G(V,E) \) is a regular graph, then \( (1, 1, \ldots, 1) \) is an eigenvector of \( A \) with eigenvalue \( n \), where \( n \) is the degree of each vertex.

Proof:

Let \( A \) be the adjacency matrix of \( G(V,E) \).

Then \( (x_1, x_2, \ldots, x_n) \) is a normalized eigenvector of \( A \) with eigenvalue \( \lambda \).

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