The representation ring of G:

$R(G)$ = formal (finite) linear combinations with integer coefficients of (finite dim, complex) representations of G (up to isomorphism), mod. relations $[V]+[W]=[V\oplus W]$

$(R(G), \oplus, \otimes)$ is a ring – the representation ring of G
↖ ↖ extend these operations to formal sums / differences of rep's by linearity!

As a set, $R(G) = \left\{ \sum_{i=1}^{k} a_i V_i \mid a_i \in \mathbb{Z} \right\}$ where V_i = the irreducible representations of G

(complete reducibility + uniqueness of decomposition into irreps.)

ie. $(R(G), +)$ is a free abelian group ($\simeq \mathbb{Z}^k$, k = # irreducibles).

Elements of $R(G)$ are called "virtual representations" (vs. genuine rep's $\sum a_i V_i$, $a_i \geq 0\ \forall i$)

Next: the character, $V \mapsto \chi_V$, can be extended by linearity to a map $R(G) \longrightarrow \underbrace{\mathbb{C}_{class}(G)}_{\text{class functions}}$. This is a ring homomorphism!
$(\chi_{U\oplus V} = \chi_U + \chi_V,\ \chi_{U\otimes V} = \chi_U \chi_V)$

The image of this map = "virtual characters" $\left(= \left\{ \sum a_i \chi_{V_i},\ a_i \in \mathbb{Z} \right\}\right)$.

Passing to complex linear combinations instead of integer ones, our results about irred. characters forming a basis say:

$$R(G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\simeq} \mathbb{C}_{class}(G) \quad \text{is an isomorphism}$$

$$\sum_{i=1}^{k} a_i [V_i] \longmapsto \chi_{\sum a_i V_i} = \sum a_i \chi_{V_i}$$

$(a_i \in \mathbb{C}$ now$)$

(tensor product of (free) $\mathbb{Z}$-modules, works same as for vector spaces).

- There are theorems of Artin and Brauer that describe the lattice of virtual characters $\Lambda = \left\{ \sum a_i \chi_{V_i},\ a_i \in \mathbb{Z} \right\}$ inside $\mathbb{C}_{class}(G)$. We'll see those after Thanksgiving.

Now we'll look at rep's of S_5 and A_5, for extra practice with characters + to motivate discussion of restriction & induction of representations

(reps of G ⟷ reps of subgroups of G).

One can start building the character table of S_5 the usual way: start with known rep's.

First we have U (trivial) and U' (alternating), and V (standard rep., dim 4).

Use: $V \oplus U \simeq$ permutation rep. $\mathbb{C}^5$, so $\chi_{V\oplus U}(\sigma) = \#\{i / \sigma(i) = i\}$, $\chi_V = \chi_{U\oplus V} - 1$.

	1	10	20	30	24	15	20
	e	(12)	(123)	(1234)	(12345)	$(12)(34)$	$(123)(45)$
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
$V' = V\otimes U'$	4	-2	1	0	-1	0	1

Then we need more. Since $|S_5| = 120 = \sum \dim^2$, we're still missing 3 irreducibles with $\sum \dim^2 = 86$; the most effective way to find them is to keep building tensor products - namely look at $V\otimes V$ (dim. 16), or rather its two pieces $\mathrm{Sym}^2 V$ (dim. 10) and $\Lambda^2 V$ (dim. 6).

$\star$ Observe: if $g : V \to V$ has eigenvalues λ_i ($g v_i = \lambda_i v_i$, $1\le i\le r$) then the corresponding map on $\mathrm{Sym}^2 V$ has eigenvalues $\lambda_i\lambda_j$, $1 \le i \le j \le r$ (recall: (v_i) basis of $V \Rightarrow (v_i v_j)$ basis of $\mathrm{Sym}^2 V$)

$\Lambda^2 V$ $\qquad\qquad\qquad\qquad$ $\lambda_i\lambda_j$, $1\le i < j \le r$ $\qquad\qquad\qquad\qquad$ $(v_i \wedge v_j)$ $\qquad$ $\Lambda^2 V$

Now, $\left.\begin{aligned} \sum_{i<j} \lambda_i\lambda_j &= \tfrac{1}{2}\left(\left(\sum \lambda_i\right)^2 - \sum \lambda_i^2\right) \\ \sum_{i\le j} \lambda_i\lambda_j &= \tfrac{1}{2}\left(\left(\sum \lambda_i\right)^2 + \sum \lambda_i^2\right)\end{aligned}\right\}$ so $\begin{aligned} \chi_{\Lambda^2 V}(g) &= \tfrac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right) \\ \chi_{\mathrm{Sym}^2 V}(g) &= \tfrac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right).\end{aligned}$

(this is true for any rep$^{\underline{n}}$).

This formula lets us calculate $\chi_{\Lambda^2 V}$ and $\chi_{\mathrm{Sym}^2 V}$ for the standard rep. of S_5.

	1	10	20	30	24	15	20
	e	(12)	(123)	(1234)	(12345)	$(12)(34)$	$(123)(45)$
V	4	2	1	0	-1	0	-1
$\Lambda^2 V$	6	0	0	0	1	-2	0
$\mathrm{Sym}^2 V$	10	4	1	0	0	2	1

Observe: $H(\chi_{\Lambda^2 V}, \chi_{\Lambda^2 V}) = \frac{1}{120}\left(6^2 + 24 + 15\cdot 2^2\right) = 1$, so $\Lambda^2 V$ is irreducible!

whereas $H(\chi_{\mathrm{Sym}^2 V}, \chi_{\mathrm{Sym}^2 V}) = \frac{1}{120}\left(\underbrace{10^2 + 10\cdot 4^2 + 20 + 15\cdot 2^2 + 20}_{360}\right) = 3$

so $\mathrm{Sym}^2 V$ splits into 3 irreducible summands.

$H(\chi_U, \chi_{\mathrm{Sym}^2 V}) = \frac{1}{120}\left(10 + 10\cdot 4 + 20 + 15\cdot 2 + 20\right) = 1 \Rightarrow$ one copy of U

similar calculations $\Rightarrow$ $\mathrm{Sym}^2 V$ also contains V with mult. 1; not U' or V'.

Hence $\mathrm{Sym}^2 V = U \oplus V \oplus W$ for some irred. 5-dim'l representation W.
Subtracting, we find χ_W – and one more, $W' = W \otimes U'$, which completes the list.

	1	10	20	30	24	15	20
	e	(12)	(123)	(1234)	(12345)	$(12)(34)$	$(123)(45)$
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
$V' = V \otimes U'$	4	-2	1	0	-1	0	1
$\Lambda^2 V$	6	0	0	0	1	-2	0
$(U \oplus V \oplus W = \mathrm{Sym}^2 V$	10	4	1	0	0	2	$1)$
W	5	1	-1	-1	0	1	1
$W' = W \otimes U'$	5	-1	-1	1	0	1	-1

<u>Remark</u>: the standard rep'n V and its exterior powers $\Lambda^2 V$, $\Lambda^3 V \simeq V'$, and $\Lambda^4 V \simeq U'$ are all irreducible! This is in fact a general property – $\forall\, 0 \le k \le n-1$, the exterior powers $\Lambda^k V$ of the standard rep. of S_n are all irreducible (see Fulton–Harris §3.2).

- Next, move on to A_5. Starting point = restrict irreducible representations of S_5 to A_5 and see which ones remain irreducible or decompose. Of course different irred. reps. of S_5 can become isomorphic after restriction – namely elements of A_5 act by id on U' so U' becomes trivial! and the restrictions of V and $V' = V \otimes U'$ become isomorphic, similarly W. The character table for S_5 gives, after restriction:

	1	20	12	12	15
	e	(123)	(12345)	(12354)	$(12)(34)$
U	1	1	1	1	1
V	4	1	-1	-1	0
W	5	-1	0	0	1
$\Lambda^2 V$	6	0	1	1	-2

Calculating $H(\chi,\chi)$ we find that U, V, W are irreducible, while $H(\chi_{\Lambda^2 V}, \chi_{\Lambda^2 V}) = 2$ so $\Lambda^2 V$ breaks into the direct sum of 2 distinct irreducibles. Also $\Lambda^2 V$ doesn't contain U, V or W, so $\Lambda^2 V = Y \oplus Z$ the last two irreducible rep's of A_5.
From $\sum \dim^2 = |A_5| = 60$ we find that $\dim Y = \dim Z = 3$. How do we find χ_Y and χ_Z?
Using orthogonality and $\chi_Y + \chi_Z = \chi_{\Lambda^2 V}$, so $\chi_Y - \chi_Z \in \mathrm{span}(\chi_U, \chi_V, \chi_W, \chi_{\Lambda^2 V})^{\perp}$

Hence $\chi_Y - \chi_Z = (0,0,a,-a,0)$, where $H(\chi_Y-\chi_Z, \chi_Y-\chi_Z)=2 \Rightarrow 24a^2=120$, $a=\pm\sqrt{5}$.

Thus:

	1 e	20 (123)	12 (12345)	12 (12354)	15 $(12)(34)$
U	1	1	1	1	1
V	4	1	-1	-1	0
W	5	-1	0	0	1
Y	3	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1
Z	3	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1

What are Y and Z ?? Recall: A_5 = rotational symmetries of an icosahedron in $\mathbb{R}^3$.
So: $A_5 \hookrightarrow SO(3) \subset GL(3,\mathbb{R}) \subset GL(3,\mathbb{C})$. ($Y$ and Z differ by an outer automorphism of A_5: conjugation by transposition inside S_5)

(The fact that the character takes irrational values implies that there does not exist a regular icosahedron (or dodecahedron) in $\mathbb{R}^3$ whose vertices all have rational coordinates! Otherwise we'd get that the representation factors through $GL(3,\mathbb{Q})$, and $tr(g)\in\mathbb{Q}$ $\forall g$)

---

* More systematic approach: if G is a finite group and $H\subset G$ a subgroup, then we have a restriction operation Res^G_H : rep^{ns} of $G \longrightarrow$ repr^{ns} of H

This is actually a functor $\mathrm{Rep}(G)\longrightarrow \mathrm{Rep}(H)$ [objects = rep of G, of H; mor = homomorphisms of rep^s]

How about the opposite direction?

Suppose V is a rep. of G, and $W\subset V$ is invariant under H (but not all of G).
Now for $g\in G$, the subspace $gW\subset V$ depends only on the coset gH, and each gW is a rep^n of gHg^{-1}, with

$$\begin{tikzcd} H \arrow[r, "\rho"] \arrow[d, "c_g", "\simeq"'] & GL(W) \arrow[d, "\text{conj. by } g"] \\ gHg^{-1} \arrow[r] & GL(gW) \end{tikzcd}$$

The simplest possible scenario is that

$$V = \bigoplus_{\sigma\in G/H} \sigma W.$$

[in general there is no reason for this to hold].

If this happens, then the rep. of G is completely determined by that of H.
Indeed, choose representatives $\sigma_1,\dots,\sigma_k\in G$ of the cosets of H (each coset $\ni$ one σ_i)
given $g\in G$, $g\sigma_i\in\sigma_j H$ for some j, so there exists $h\in H$ s.t. $g=\sigma_j h\sigma_i^{-1}$.
Then g acts by mapping $\sigma_i W$ to $\sigma_j W$, with $g(\sigma_i w)=\sigma_j h(w)$.

(Remark: $\dim V = |G/H|\cdot \dim W$).

Def: A representation V of G, with a subspace $W \subset V$ which is invariant under the subgroup $H \subset G$ (ie. a subrep. of $\mathrm{Res}^G_H V$), is said to be <u>induced</u> by $W \in \mathrm{Rep}(H)$ if, as a vector space, $V = \bigoplus_{\sigma \in G/H} \sigma W$. Write $V = \mathrm{Ind}^G_H W$.

ie. fixing one element in each coset, $\sigma_1, \dots, \sigma_k \in G$, we can write each $v \in V$ uniquely as $v = \sigma_1 w_1 + \dots + \sigma_k w_k$ for $w_1, \dots, w_k \in W$.

Thm: Given a representation W of H, the induced representation $V = \mathrm{Ind}^G_H W$ exists and is unique up to isomorphism of G-rep^s

Pf:
- Uniqueness: given $V \in \mathrm{Rep}(G)$ and $W \subset V$ invariant under H & s.t. $V = \bigoplus_{i=1}^k \sigma_i W$, necessarily $g \in G$ acts by mapping $\sigma_i W$ to $\sigma_j W$, where j is such that $g\sigma_i \in \sigma_j H$, ie. $h = \sigma_j^{-1} g \sigma_i \in H$, and necessarily $g(\sigma_i w) = \sigma_j h w \in \sigma_j W$. This determines the G-action uniquely.
- Existence: build $V = \bigoplus_{i=1}^k \sigma_i W$ where the σ_i are now formal symbols (ie. the direct sum of $k = |G/H|$ copies of W), and make $g \in G$ act as above. $\square$.

Examples: 1) The permutation rep. associated to the left action of G on G/H is induced by the trivial representation of H. Indeed V has a basis $\{e_\sigma\}_{\sigma \in G/H}$; the basis element e_H (for the coset H) is fixed by H, so $W = \mathrm{span}(e_H)$ is invariant under H, and $gW = \mathrm{span}(e_{gH})$, with

$$V = \bigoplus_{gH \in G/H} \mathrm{span}(e_{gH}) = \bigoplus_{gH \in G/H} gW.$$

2) The regular rep. of G is induced by the regular rep. of H: here $W = \mathrm{span}\{e_h, h \in H\} \subset V = \mathrm{span}\{e_g, g \in G\}$.