6.5084 / 18.5096

LINEAR ALGEBRA AND OPTIMIZATION

TODAY: Monday 9/21

- Pset Due TODAY 9PM
- Midterm Wed Oct 7
- Check-ups every lecture
- Pizza, Office Hours
Challenge: Find the dimension of the subspace spanned by
\[\left\{ \begin{pmatrix} \frac{2}{3} \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right\}\]

4 vectors in \(\mathbb{R}^3\)

They span a \(2\)-dim subspace!

Can compute, e.g., using Gaussian elim.

Easy proof: All vectors contained in plane \(2x - y - z = 0\).
Lecture 8

The RANK of a matrix

Last time:

- Linear independence
- Generators
- Bases
- Dimension
Subspaces:

Two usual descriptions:

- **Generators**
  \[ S = \text{span}\{v_1, \ldots, v_k\} \]

  \[ S = \text{span}\{(1,0,0), (0,1,0), (0,0,1)\} \]

- **Equations**
  \[ S = \{ x \mid a_i \cdot x = 0 \} \]
  \[ \{(x,y,z) \in \mathbb{R}^3 : x = y, y = z, x = z\} \]

**BASES** are very useful.

- But, where do they come from?
- How to compute them?

**Claim:** If we have generators, can easily "refine" them to a basis.
**TASK:**

Given subspace \( S = \text{span} \{v_1, \ldots, v_n\} \)

Compute basis \( B \) of \( S \).

**Algorithm** (basic idea)

Initialize basis: \( B = \{\} \)

for \( k \) from 1 to \( n \)

if \( v_k \) is not in \( \text{span}(B) \)

\[ B = B \cup \{v_k\} \quad (\text{add } v_k \text{ to basis}) \]

end for

return \( B \).

**Claim:**

At every step, \( B \) is a basis of \( \text{span} \{v_1, \ldots, v_k\} \).
We already have an implementation!
**Column Rank:** of \( A \)

\[
\begin{bmatrix}
V_1 & \cdots & V_n
\end{bmatrix}
\]

\[0 \leq \text{colrank}(A) \leq n\]

**Row Rank:**

\[
A = \begin{bmatrix}
a_1 \\
\vdots \\
a_m
\end{bmatrix}
\]

\[0 \leq \text{rowrank}(A) \leq m\]
Claim: For any matrix $A$

Row rank $(A) = \text{Column rank} (A) = \text{Rank} (A)$

Def: The rank of a matrix $A$ is the number of pivots.
\[ A = \begin{bmatrix} -1 & 2 & 2 & 1 \\ 0 & 0 & 3 & 2 \\ -2 & 4 & 1 & 0 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 2 \end{bmatrix} \]

\[ \text{rank}(A) = \# \text{ pivots} \]

and

- Pivot columns of \( A \) are LI
- Pivot columns of \( R \) are LI
- Pivot rows of \( R \) are LI

\[ A \ldots \quad (\text{no!}) \]

Example:

\[ A = \begin{bmatrix} 0 & -\frac{16}{15} & -\frac{4}{15} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]

\[ E = \begin{bmatrix} 0 & \frac{1}{3} & 1 \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \]

\[ \Rightarrow \quad \text{rank}(A) = 2 \]
RANK AND FACTORIZATIONS:

\[
\begin{pmatrix}
\text{m} \\
\text{n}
\end{pmatrix}
\begin{pmatrix}
\text{k} \\
\text{n}
\end{pmatrix}
\]

\[
A = BC
\]

**Rank:** minimum \( k \) for which such factorization exists.

**Why do we care?**

- Efficient algorithms (e.g., MatMult).
- Dimensionality reduction
- Recommendation systems (e.g., Netflix).
Let $A$ be a $m \times n$ matrix ($m \geq n$, "tall matrix")

Then: The following are equivalent:

- The columns of $A$ are linearly independent
- $Ax = b$ is solvable, has a unique solution
- $A$ has a left inverse
- $N(A) = \{0\}$
- $\text{rank}(A) = n$ ("full column rank")
Let $A$ be a $m \times n$ matrix (when $m \leq n$, "fat" matrix) with $m$ rows

$$\begin{bmatrix}
\vdots \\
\vdots
\end{bmatrix}$$

$n$ columns

$[a_{ij}]$

**Thm:** The following are equivalent:

- The rows of $A$ are linearly independent
- $Ax = b$ is solvable for every $b$
- $A$ has a right inverse
- $\mathcal{C}(A) = \mathbb{R}^m$
- $\text{rank}(A) = m$ ("full row rank")
Let $A$ be a square $n \times n$ matrix

$\begin{bmatrix} a_{ij} \end{bmatrix}$

Thm: The following are equivalent:

- $A$ is invertible (non-singular) $A^{-1}$ exists
- $Ax = b$ always has a unique solution
- $N(A) = \{0\}$
- $C(A) = \mathbb{R}^n$
- $\text{rank}(A) = n$
More generally: \[ A \in \mathbb{R}^{m \times n} \]

"Rank-Nullity" Theorem.

\[ \text{rank}(A) + \dim N(A) = n \]

(more about this later in the course)
PUZZLE!

Alice “proves”:

**A has a left inverse** $\implies A x = b$ is always solvable

*(false!)*

\[A x = b \implies N A x = N b \implies [x = N b] \]

\[\neg A \]

“**Proof**”:

$\exists N: NA = I$ (exist left inv)

\[
A x = b \implies N A x = N b \implies [x = N b]
\]

Q: What’s wrong with this???