Rate of Change and Slope

1 Rate of Change

Rate of change is a ratio that describes, on average, how much one quantity changes with respect to a change in another quantity.

Key Concept: Rate of Change

If \( x \) is the independent variable and \( y \) is the dependent variable, then

\[
\text{rate of change} = \frac{\text{change in } y}{\text{change in } x}
\]

Real-World Example 1: Find Rate of Change

ENTERTAINMENT Use the table to find the rate of change. Then explain its meaning.

<table>
<thead>
<tr>
<th>Number of Computer Games</th>
<th>Total Cost ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>78</td>
</tr>
<tr>
<td>4</td>
<td>156</td>
</tr>
<tr>
<td>6</td>
<td>234</td>
</tr>
</tbody>
</table>

The rate of change is \( \frac{39}{1} \). This means that each game costs $39.

Guided Practice

1. REMODELING The table shows how the tiled surface area changes with the number of floor tiles.
   - A. Find the rate of change.
   - B. Explain the meaning of the rate of change.

<table>
<thead>
<tr>
<th>Number of Floor Tiles</th>
<th>Area of Tiled Surface (ft²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>48</td>
</tr>
<tr>
<td>6</td>
<td>96</td>
</tr>
<tr>
<td>9</td>
<td>144</td>
</tr>
</tbody>
</table>

So far, you have seen rates of change that are constant. Many real-world situations involve rates of change that are not constant.

Real-World Example 2: Compare Rates of Change

AMUSEMENT PARKS The graph shows the number of people who visited U.S. theme parks in recent years.


\[
\begin{align*}
\text{change in attendance} & = 324 - 317 = 7 \\
\text{change in time} & = 2002 - 2000 = 2 \\
\text{rate of change} & = \frac{7}{2} = 3.5
\end{align*}
\]

Over this 2-year period, attendance increased by 7 million, for a rate of change of 3.5 million per year.

b. Explain the meaning of the rate of change in each case.

For 2000–2002, on average, 3.5 million more people went to a theme park each year than the last.

For 2002–2004, on average, 0.5 million more people attended theme parks each year than the last.

c. How are the different rates of change shown on the graph?

There is a greater vertical change for 2000–2002 than for 2002–2004. Therefore, the section of the graph for 2000–2002 is steeper.

Guided Practice

2. Refer to the graph above. Without calculating, find the 2-year period that has the least rate of change. Then calculate to verify your answer.

A rate of change is constant for a function when the rate of change is the same between any pair of points on the graph of the function. Linear functions have a constant rate of change.
Determine whether each function is linear. Explain.

**Example 3: Constant Rates of Change**

StudyTip
Linear or Nonlinear Function? Notice that the changes in $x$ and $y$ are not the same. For the rate of change to be linear, the change in $x$-values must be constant and the change in $y$-values must be constant.

| $x$ | $y$ | Rate of change
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-6</td>
<td>$-6 - (-4)$ or $-2$</td>
</tr>
<tr>
<td>4</td>
<td>-8</td>
<td>$-8 - (-6)$ or $-2$</td>
</tr>
<tr>
<td>7</td>
<td>-10</td>
<td>$-10 - (-8)$ or $-2$</td>
</tr>
<tr>
<td>10</td>
<td>-12</td>
<td>$-12 - (-10)$ or $-2$</td>
</tr>
<tr>
<td>13</td>
<td>-14</td>
<td>$-14 - (-12)$ or $-2$</td>
</tr>
</tbody>
</table>

Thus, the function is linear.

| $x'$ | $y'$ | Rate of change
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>11</td>
<td>$11 - (-4)$ or 15</td>
</tr>
<tr>
<td>-2</td>
<td>15</td>
<td>$15 - 11$</td>
</tr>
<tr>
<td>-1</td>
<td>19</td>
<td>$19 - 15$</td>
</tr>
<tr>
<td>1</td>
<td>23</td>
<td>$23 - 19$</td>
</tr>
<tr>
<td>2</td>
<td>27</td>
<td>$27 - 23$</td>
</tr>
</tbody>
</table>

This rate of change is not constant. Thus, the function is not linear.

**GuidedPractice**

3A. $x$ | $y$
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>11</td>
</tr>
<tr>
<td>-2</td>
<td>15</td>
</tr>
<tr>
<td>-1</td>
<td>19</td>
</tr>
<tr>
<td>1</td>
<td>23</td>
</tr>
<tr>
<td>2</td>
<td>27</td>
</tr>
</tbody>
</table>

3B. $x$ | $y$
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>11</td>
</tr>
<tr>
<td>-2</td>
<td>-4</td>
</tr>
<tr>
<td>-1</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

**Find Slope** The slope of a nonvertical line is the ratio of the change in the $y$-coordinates (rise) to the change in the $x$-coordinates (run) as you move from one point to another.

It can be used to describe a rate of change. Slope describes how steep a line is. The greater the absolute value of the slope, the steeper the line.

The graph shows a line that passes through $(-1, 3)$ and $(2, -2)$.

\[
slope = \frac{rise}{run} = \frac{change \ in \ y-coordinates}{change \ in \ x-coordinates} = \frac{-2 - 3}{2 - (-1)} or \frac{-5}{3}
\]

So, the slope of the line is $-\frac{5}{3}$.

Because a linear function has a constant rate of change, any two points on a nonvertical line can be used to determine its slope.

**GuidedPractice**

Find the slope of the line that passes through each pair of points.

4A. $(3, 6), (4, 8)$
4B. $(-4, -2), (0, -2)$
4C. $(-4, 2), (-2, 10)$
4D. $(6, 7), (-2, 7)$
4E. $(4, 3), (-1, 11)$
### Example 5: Undefined Slope

Find the slope of the line that passes through \((-2, 4)\) and \((-2, -3)\).

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 4}{-2 - (-2)} = \frac{-7}{0} \text{ or undefined}
\]

The graph of the line is horizontal. If the change in \(x\)-values is 0, then the graph of the line is horizontal. If the change in \(x\)-values is 0, then the slope is undefined. This graph is a vertical line.

### Guided Practice

Find the slope of the line that passes through each pair of points.

5A. \((6, 3), (6, 7)\)  
5B. \((-3, 2), (-3, -1)\)

The graphs of lines with different slopes are summarized below.

### Concept Summary: Slope

<table>
<thead>
<tr>
<th>Positive Slope</th>
<th>Negative Slope</th>
<th>Slope of 0</th>
<th>Undefined Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line slopes up from left to right</td>
<td>Line slopes down from left to right</td>
<td>Horizontal line</td>
<td>Vertical line</td>
</tr>
<tr>
<td>The function values are increasing over the entire domain</td>
<td>The function values are decreasing over the entire domain</td>
<td>The function values are constant over the entire domain</td>
<td>The relation is not a function</td>
</tr>
</tbody>
</table>

### Example 6: Find Coordinates Given the Slope

Find the value of \(r\) so that the line through \((1, 4)\) and \((-5, r)\) has a slope of \(\frac{3}{2}\).

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{r - 4}{-5 - 1} = \frac{3}{2}
\]

\[
3(r - 4) = 12
\]

So, the line goes through \((-5, 2)\).

### Guided Practice

Find the value of \(r\) so that the line that passes through each pair of points has the given slope.

6A. \((-2, 6), (r, -4); m = -5\)  
6B. \((r, -6), (5, -8); m = -8\)
Example 1
Find the rate of change represented in each table or graph.

16. [Graph]

17. [Graph]

Example 2
18. SPORTS What was the annual rate of change from 2004 to 2008 for women participating in collegiate lacrosse? Explain the meaning of the rate of change.

Year | Number of Women |
-----|-----------------|
2004 | 5545            |
2008 | 6830            |

19. RETAIL The average retail price in the spring of 2009 for a used car is shown in the table at the right.
a. Write a linear function to model the price of the car with respect to age.
b. Interpret the meaning of the slope of the line.
c. Assuming a constant rate of change predict the average retail price for a 7-year-old car.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Value ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>17,317</td>
</tr>
<tr>
<td>3</td>
<td>16,127</td>
</tr>
</tbody>
</table>

Example 3
20. Determine whether each function is linear. Write yes or no. Explain.

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{ } & 4 & 2 & 0 & -2 & -4 \\
\hline
f(x) & 1 & 1 & 3 & 5 & 7 \\
\end{array}
\]

21. [Table]

22. [Table]

Example 4-6 Find the slope of the line that passes through each pair of points.

24. (4, 3), (−1, 6)  
25. (8, −2), (1, 1)  
26. (2, 2), (−2, −2)  
27. (6, −10), (6, 14)  
28. (5, −4), (9, −4)  
29. (11, 73), (−6, 2)  
30. (−3, 5), (3, 6)  
31. (−3, 2), (7, 2)  
32. (8, 10), (−4, −6)  
33. (−8, 6), (−6, 4)  
34. (−12, 15), (18, −13)  
35. (−8, −15), (−2, 5)

Example 5
Find the value of \( r \) so the line that passes through each pair of points has the given slope.

36. (12, 10), (−2, 7), \( m = -4 \)  
37. (−5, 1), (3, 13), \( m = 8 \)  
38. (3, 5), (−3, 1), \( m = \frac{1}{4} \)  
39. (−2, 8), (7, 4), \( m = -\frac{1}{2} \)

Example 6
Find the rate of change of \( r \) so the line that passes through each pair of points has the given slope.

40. [Graph]

Example 7
41. [Graph]

42. DRIVING When driving up a certain hill, you rise 15 feet for every 1000 feet you drive forward. What is the slope of the road?

Find the slope of the line that passes through each pair of points.

43. \[
\begin{array}{c|c|c}
\text{x} & \text{y} \\
\hline
4.5 & 1 \\
5.3 & 2 \\
\end{array}
\]

44. \[
\begin{array}{c|c|c}
\text{x} & \text{y} \\
\hline
0.75 & 1 \\
0.75 & 1 \\
\end{array}
\]

45. \[
\begin{array}{c|c|c}
\text{x} & \text{y} \\
\hline
-2 & 1 \\
-2 & 1 \\
\end{array}
\]

46. MULTIPLE REPRESENTATIONS In this problem, you will investigate why the slope of a line through any two points on that line is constant.

a. Visual Sketch a line \( t \) that contains points \( A, B, A', \) and \( B' \) on a coordinate plane.

b. Geometric Add segments to form right triangles \( ABC \) and \( A'B'C' \) with right angles at \( C \) and \( C' \). Describe \( \triangle AC \) and \( \triangle A'C' \), and \( \triangle BC \) and \( \triangle B'C' \).

c. Verbal How are triangles \( ABC \) and \( A'B'C' \) related? What does that imply for the slope between any two distinct points on line \( t \)?

47. BASKETBALL The table below shows the average points per game (PPG) Michael Redd has scored in each of his first 9 seasons with the NBA's Milwaukee Bucks.

<table>
<thead>
<tr>
<th>Season</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>PPG</td>
<td>22</td>
<td>11.4</td>
<td>16.3</td>
<td>21.7</td>
<td>23.0</td>
<td>25.4</td>
<td>26.7</td>
<td>22.7</td>
<td>21.2</td>
</tr>
</tbody>
</table>

a. Make a graph of the data. Connect each pair of adjacent points with a line.
b. Use the graph to determine in which period Michael Redd's PPG increased the fastest. Explain your reasoning.
c. Discuss the difference in the rate of change from season 1 through season 4, from season 4 through season 7, from season 7 through season 9.

H.O.T. Problems Use Higher-Order Thinking Skills

48. REASONING Why does the Slope Formula not work for vertical lines? Explain.

49. OPEN ENDED Use what you know about rate of change to describe the function represented by the table.

50. CHALLENGE Find the value of \( a \) so the line that passes through \( (a, b) \) and \( (c, d) \) has a slope of \( \frac{1}{2} \).

51. WRITING IN MATH Explain how the rate of change and slope are related and how to find the slope of a line.

52. CRITICAL THINKING Kyle and Luna are finding the value of \( a \) so the line that passes through \( (10, a) \) and \( (−2, 8) \) has a slope of \( \frac{1}{2} \). Is either of them correct? Explain.