If G is a finite group and $H \subset G$ a subgroup, then we have a restriction functor $\mathrm{Res}^G_H : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(H)$. In the opposite direction, how do rep's of H give rep's of G?

Answer: induced representations = rep. of G built from $|G/H|$ many copies of a rep. of H, assembled into a G-rep. according to the manner in which left mult by $g \in G$ acts on cosets of H.

Def: A representation V of G, with a subspace $W \subset V$ which is invariant under the subgroup $H \subset G$ (ie. a subrep. of $\mathrm{Res}^G_H V$), is said to be induced by $W \in \mathrm{Rep}(H)$ if, as a vector space, $V = \bigoplus_{\sigma \in G/H} \sigma W$. Write $V = \mathrm{Ind}^G_H W$.

ie. fixing one element in each coset, $\sigma_1, \dots, \sigma_k \in G$, we can write each $v \in V$ uniquely as $v = \sigma_1 w_1 + \dots + \sigma_k w_k$ for $w_1, \dots, w_k \in W$.

Thm: Given a representation W of H, the induced representation $V = \mathrm{Ind}^G_H W$ exists and is unique up to isomorphism of G-rep^s

Pf:
- Uniqueness: given $V \in \mathrm{Rep}(G)$ and $W \subset V$ invariant under H & s.t. $V = \bigoplus_{i=1}^k \sigma_i W$, necessarily $g \in G$ acts by mapping $\sigma_i W$ to $\sigma_j W$, where j is such that $g\sigma_i \in \sigma_j H$, ie. $h = \sigma_j^{-1} g \sigma_i \in H$, and necessarily $g(\sigma_i w) = \sigma_j h w \in \sigma_j W$. This determines the G-action uniquely.
- Existence: build $V = \bigoplus_{i=1}^k \sigma_i W$ where the σ_i are now formal symbols (ie. the direct sum of $k = |G/H|$ copies of W), and make $g \in G$ act as above. □.

(Note: by construction, $\dim V = |G/H| \cdot \dim W$).

Examples: 1) The permutation rep. associated to the left action of G on G/H is induced by the trivial representation of H. Indeed V has a basis $\{e_\sigma\}_{\sigma \in G/H}$; the basis element e_H (for the coset H) is fixed by H, so $W = \mathrm{span}(e_H)$ is invariant under H, and $gW = \mathrm{span}(e_{gH})$, with
$$V = \bigoplus_{gH \in G/H} \mathrm{span}(e_{gH}) = \bigoplus_{gH \in G/H} gW.$$

2) The regular rep. of G is induced by the regular rep. of H: here $W = \mathrm{span}\{e_h, h \in H\} \subset V = \mathrm{span}\{e_g, g \in G\}$.

- Fact: $\mathrm{Ind}^G_H(W \oplus W') = \mathrm{Ind}^G_H(W) \oplus \mathrm{Ind}^G_H(W')$, but $\mathrm{Ind}(W \otimes W') \neq \mathrm{Ind}(W) \otimes \mathrm{Ind}(W')$.

On the other hand, if U is a rep. of G and W a rep. of H, then

$$\mathrm{Ind}(\mathrm{Res}(U)\otimes W) = U \otimes \mathrm{Ind}(W).$$

(indeed: $\mathrm{Ind}(W) = \bigoplus_{\sigma\in G/H} \sigma W$, so $U\otimes \mathrm{Ind}(W) = \bigoplus_{\sigma\in G/H}(U\otimes \sigma W) = \bigoplus_{G/H} \sigma(U\otimes W)$,

where $U\otimes W \subset U\otimes \mathrm{Ind}(W)$ is invariant under H and $\simeq \mathrm{Res}(U)\otimes W$ as H-rep^{ns}).

<u>in particular</u>: $$\mathrm{Ind}(\mathrm{Res}(U)) = U\otimes \mathrm{Ind}(\text{trivial}) = U\otimes (\text{permut. rep. } G/H).$$

- We can actually calculate the character of an induced representation!

Choose representatives $\sigma_1,\dots,\sigma_k$ of cosets of H as usual; $g\in G$ maps $\sigma_i W$ to $\sigma_j W$ s.t. $g\sigma_i \in \sigma_j H$. If $i\neq j$ then this doesn't contribute to $\mathrm{tr}(g)$. If $i=j$ then $h=\sigma_i^{-1} g\sigma_i \in H$ and g maps $\sigma_i W$ to itself by $g(\sigma_i w) = \sigma_i h w$, so $\mathrm{tr}(g_{|\sigma_i W}) = \mathrm{tr}(h_{|W}) = \chi_W(h)$.

Summing over σ_i's: $$\chi_{\mathrm{Ind}\,W}(g) = \sum_{\substack{\sigma_i\in G/H \\ \text{st. } \sigma_i^{-1}g\sigma_i\in H}} \chi_W(\sigma_i^{-1} g\sigma_i) = \frac{1}{|H|}\sum_{\substack{s\in G\\ \text{st. } s^{-1}gs\in H}} \chi_W(s^{-1}gs).$$

- A key property for understanding induced representations is <u>Frobenius reciprocity</u>

<u>Prop</u>: If U is a representation of G, and W a rep. of H, then every H-equivariant map $W\to \mathrm{Res}(U)$ extends uniquely to a G-equivariant map $\mathrm{Ind}(W)\to U$:
$$\mathrm{Hom}_H(W,\mathrm{Res}(U)) \simeq \mathrm{Hom}_G(\mathrm{Ind}(W),U).$$

<u>Proof</u>: Choose representatives $\sigma_1,\dots,\sigma_k\in G$ of the cosets of H, and let $V=\mathrm{Ind}(W)=\bigoplus \sigma_i W$: given $\varphi: W\to \mathrm{Res}(U)$ H-equivariant, if $\tilde\varphi: V\to U$ is G-equivariant and $\tilde\varphi_{|W}=\varphi$, then necessarily we have a comm. diagram

$$\begin{tikzcd} W \arrow[r, "\varphi"] \arrow[d, "\sigma_i"'] & U \arrow[d, "\sigma_i"] \\ \sigma_i W \arrow[r, "\tilde\varphi"] & U \end{tikzcd}$$

ie. $\tilde\varphi_{/\sigma_i W}$ is given by $\tilde\varphi(\sigma_i w) = \sigma_i\cdot\varphi(w)$

the action of $\sigma_i\in G$ on $\varphi(w)\in U$.

This determines $\tilde\varphi$ uniquely.

To check $\tilde\varphi$ is G-equivariant, recall $g\in G$ acts on V by mapping $\sigma_i W$ to $\sigma_j W$ st. $g\sigma_i = \sigma_j h\in\sigma_j H$, via $g(\sigma_i w) = \sigma_j\cdot hw$. Given $\sigma_i w\in \sigma_i W$,

$$\tilde\varphi(g(\sigma_i w)) = \tilde\varphi(\sigma_j\cdot hw) = \underset{\substack{\uparrow\\ \text{acting on } U}}{\sigma_j}\cdot\varphi(hw) = \sigma_j h\,\varphi(w) = g\sigma_i\,\varphi(w) = g(\tilde\varphi(\sigma_i w)).$$

$\Rightarrow \tilde\varphi g = g\tilde\varphi$ on $\sigma_i W$ $\forall i$, hence on V.

So: φ does have a unique G-equivariant extension $\tilde\varphi$.

Conversely, given $\tilde\varphi\in\mathrm{Hom}_G(V,U)$, $\tilde\varphi$ is H-equivariant, and hence its restriction to $W\subset V$ is H-equivariant. $\square$

Comparing dimensions, $\dim \mathrm{Hom}_G(\dots) = \dim \mathrm{Hom}_H(\dots) \Rightarrow$

<u>Corollary</u> (<u>Frobenius reciprocity</u>): $\langle \chi_{\mathrm{Ind}\, W}, \chi_U \rangle_G = \langle \chi_W, \chi_{\mathrm{Res}\, U} \rangle_H$.

Thus: if U is an irred. rep. of G and W an irred. rep of H, then the number of times W appears in $\mathrm{Res}(U)$ is equal to the number of times U appears in $\mathrm{Ind}(W)$.

<u>Example</u>: $G = S_4 \supset H = S_3$: restrictions of irred. reps of S_4 are

- trivial: $\mathrm{Res}(U_4) = U_3$
- alternating: $\mathrm{Res}(U_4') = U_3'$
- standard: $\mathrm{Res}(V_4) = V_3 \oplus U_3$
  (since permut. rep. $\mathbb{C}^4$ restricts to permut. $\oplus$ trivial: $\mathrm{Res}(V_4 \oplus U_4) = V_3 \oplus U_3 \oplus U_3$).
- $V_4' = V_4 \otimes U_4'$: $\mathrm{Res}(V_4') = V_3 \oplus U_3'$ (using $V_3 \otimes U_3' \simeq V_3$).
- W (factors through $S_4/\{(ij)(kl)\} \simeq S_3$): $\mathrm{Res}(W) = V_3$.

(or instead of arguing explicitly, one can just use character tables!).

So by Frobenius reciprocity, $\mathrm{Ind}(V_3) = \bigoplus$ of the irred. reps of S_4 whose restrictions contain V_3

(this has dim. $8 = 4 \cdot 2$) $= V_4 \oplus V_4' \oplus W$.

<u>Another example</u>. $H = \langle (1234) \rangle \simeq \mathbb{Z}/4 \subset G = S_4$:

the irred. reps of H are 1-dimensional, with (1234) acting by a power of i

U_0 = trivial, U_1, U_2, U_3: (1234) acts by $i, i^2 = -1, i^3 = -i$.

To find the induced representations, look at irreds. of S_4:

	1 e	6 (12)	8 (123)	6 (1234)	3 $(12)(34)$
U	1	1	1	1	1
U'	1	−1	1	−1	1
V	3	1	0	−1	−1
V'	3	−1	0	1	−1
W	2	0	−1	0	2

and the eigenvalues of (1234) & $(1234)^2 = (13)(24)$

$U \mapsto U_0$

$U' \mapsto U_2$

$V \mapsto U_1 \oplus U_2 \oplus U_3 \leftarrow \begin{cases} \chi((13)(24)) = -1 \Rightarrow \lambda_i^2 = -1, -1, +1 \\ \chi(1234) = -1 \Rightarrow \lambda_i = i, -i, -1 \end{cases}$

$V' \mapsto U_3 \oplus U_0 \oplus U_1$

$W \mapsto U_0 \oplus U_2$

<u>or</u>: $U \oplus V$ has $(1234) \mapsto \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & 1 & \ddots & \\ & & 1 & 0 \end{pmatrix}$ eigenvalues $\pm 1, \pm i$, so $\mathrm{Res}(U \oplus V) = U_0 \oplus \dots \oplus U_3$.

Frobenius reciprocity $\Rightarrow$

$\mathrm{Ind}(U_0) = U \oplus V' \oplus W$ (= the permutation rep. of S_4 on G/H).

$\mathrm{Ind}(U_1) = \mathrm{Ind}(U_3) = V \oplus V'$

$\mathrm{Ind}(U_2) = U' \oplus V \oplus W$ ($= U' \otimes \mathrm{Ind}(U_0)$, consistent with $U_2 = \mathrm{Res}(U')$).

* Some of the key motivation for studying induced representations comes from two deep theorems of Artin & Brauer

Thm (Artin) Every character of a representation of G is a linear combination with rational coefficients of characters of representations induced from cyclic subgroups of G.

Thm (Brauer) Every character of a representation of G is a linear combination with integer coefficients of characters of representations induced from "elementary" subgroups of G.

where elementary = isomorphic to a product $C \times H$, H p-group $|H| = p^k$
C cyclic $\simeq \mathbb{Z}/n$, $p \nmid n$.

(won't prove. See eg. Serre's "Reps. of finite groups")

Real representations: we've studied actions of finite groups on complex vector spaces, now we want to do the same for real ones.

We have a map $\{\text{real rep}^{\text{ns}}\ V_0\} \longrightarrow \{\text{complex rep}^{\text{ns}}\}$

$$V_0 \longmapsto V = V_0 \otimes_{\mathbb{R}} \mathbb{C} = V_0 \oplus i V_0.$$

(G acts by $g(v+iw) = gv + i\,gw$).

Def: A complex rep. V of G is called real if there exists a rep. over $\mathbb{R}$, V_0, st. $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$

Necessary condition: χ_V must take real values! This is alas not a sufficient cond$^{\underline{n}}$

Ex: the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, $i^2 = j^2 = k^2 = ijk = -1$ acts on $\mathbb{C}^2$ by

$$\pm 1 \mapsto \pm \mathrm{Id}, \quad \pm i \mapsto \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \pm j \mapsto \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pm k \mapsto \pm\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$\chi(\pm 1) = \pm 2$, all others have $\chi = 0$: so χ takes real values.

However this does not come from a 2-dimensional real representation: $Q \not\hookrightarrow GL(2,\mathbb{R})$.
(This is because a real representation of a finite group has an invariant inner product, by the same averaging trick as in the complex case, so we'd get $Q \hookrightarrow O(2)$, with -1 acting by $-\mathrm{Id}$, but only 2 elements of $O(2)$ square to $-\mathrm{Id}$ (rotations by $\pm 90°$) while we need 6 such elements for $\pm i, \pm j, \pm k$.)

If V_0 is a representation of G over $\mathbb{R}$, then it has an invariant inner product $\langle , \rangle$. Extending, this yields a nondegenerate symmetric bilinear form on $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$. We'll see:

Thm: An irreducible complex representation V of a finite group G is real iff V carries a G-invariant nondegenerate symmetric bilinear form.