Physical Sciences 2: Assignments for Oct. 24 - Oct 31
Homework #7: Elasticity and Fluid Statics
Solution Key

After completing this homework, you should…

- Be able to describe what is meant by elasticity and how it relates to Hooke’s law
- Be able to explain why the Young’s modulus is necessary and why Hooke’s law cannot always be used
- Understand the stress-strain formula and be able to explain what all terms mean
- Be able to interpret a stress-strain plot
- Be able to explain the static properties of fluids: density, compressibility, and pressure
- Understand Pascal’s principle and be able to apply it
- Be able to explain how and why pressure in a fluid varies and the consequences (buoyancy)
- Know Archimedes’ Principle and be able to apply it to solve problems involving buoyant forces
Here are summaries of this module’s important concepts to help you complete this homework:

Module 7: Elasticity and Fluid Statics
Compiled by Kristina Callaghan

**Elasticity**
- Any material can stretch due to its spring-like molecular bonds.
  - Elastic limit: maximum $\Delta L$ for which the material will return to initial length $L$; past this point, the object will permanently deform and/or break.
- The relationship between the force applied and the change in distance is:
  \[ \frac{F}{A} = Y \cdot \frac{\Delta L}{L} \]
  - This has the same form as Hooke’s law:
    \[ F = \frac{(Y \cdot A)}{L} \Delta L \]
  - Positive RHS because $F$ and $\Delta L$ in same direction.
  - Strain is the quantity $F/A$.
  - Strain is the quantity $\Delta L/L$.

**Fluid Statics**
- Density $\rho$ is the ratio of mass $m$ and volume $V$, and is an intrinsic property of the material:
  \[ \rho = \frac{m}{V} \]
  - Density of water: $\rho_{\text{water}} = 1.0\text{g/cm}^3 = 1000 \text{ kg/m}^3$.
- A fluid is incompressible, meaning its molecules are already as close to each other as possible without touching.
- Pressure $P$ is the amount of force exerted over some cross-sectional area $A$:
  \[ P = \frac{F}{A} \]
  - Units: 1 Pascal $(Pa) = 1 \text{N/m}^2$; 1 atm $= 101.3 \text{kPa}$; 1 mm Hg $= 133.3 \text{ Pa}$ (atmosphere) / 760 mm Hg (torr of mercury).
Fluid Statics

- **Pascal's Principle**: pressure is transmitted undiminished in an enclosed static fluid

Example: consider an object at depth \( h \); above the liquid is some pressure \( P_0 \)

\[
P_i = P_0 + \Delta P \cdot g \cdot h
\]

The pressure at depth \( h \), denoted as \( P_i \), is equal to the pressure \( P_0 \) above the liquid, plus the static fluid pressure \( \Delta P \cdot g \cdot h \)

- This means that the pressure is the same at all points on a horizontal line in a fluid (all of these points are at the same depth).

- **Archimedes Principle**: a fluid exerts an upward buoyant force \( F_B \) on an object in the fluid

  - magnitude of \( F_B \) equal to the weight of the fluid displaced by an object

\[
F_B = \frac{M_{\text{displaced}}}{\text{fluid}} \cdot g
\]

\[
F_B = (\rho_{\text{fluid}} \cdot V_{\text{displaced}}) \cdot g
\]
1. Let it Fall (1 pt) Explicitly show that an object of density $\rho$ will sink when submerged in a fluid of density $\rho_{\text{fluid}}$, where $\rho_{\text{fluid}} < \rho$, and find its acceleration. Assume the impact of the drag force is negligible.

From our free-body diagram, we can write out Newton’s 2nd Law to find the acceleration of the object,

$$\sum_i F_{y,i} = F_B - F_g = ma_y$$

$$\sum_i F_{y,i} = \rho_{\text{fluid}} Vg - \rho Vg = \rho Vay \rightarrow \left(\frac{\rho_{\text{fluid}} - \rho}{\rho}\right)g = ay$$

Since $\rho_{\text{fluid}} / \rho < 1$, we see that the acceleration in the y-direction will be negative,

$$ay = -\left(1 - \frac{\rho_{\text{fluid}}}{\rho}\right)g$$

2. Hanging by a Thread (2 pts). A spider spins a fine thread of spider silk between two posts a distance $L_0$ apart. The thread has a diameter of $d = 5 \times 10^{-6} \text{ m}$ and an equilibrium length of $L_0$. When the spider (with mass $m = 2$ milligram) hangs from the center of the thread, the thread sags and stretches to support the weight of the spider, making an angle of $\theta = 10^\circ$ below the horizontal, as shown.

a) Why must the thread sag when the spider hangs from it?

The thread must sag due to an analysis of Newton’s 2nd law for the spider. If the spider is at rest, there must be some force in the up direction to counteract the spider’s weight (acting downwards), such that the net force is zero. This must come from the thread; upon sagging there is now a component of the thread’s tension acting in the up/down direction.
b) Calculate the tensile strain $\varepsilon$ in the thread and the Young's modulus $Y$ for the silk.

The tensile strain is defined as $\varepsilon = \frac{\Delta L}{L_0} = \frac{L - L_0}{L_0}$, so we need to find an expression for $L$. The thread stretches from an original length $L_0$ to a final length of $L$. By looking at the geometry we can write $L$ in terms of $L_0$:

$$\frac{L_0}{2} = \frac{L}{2} \cos(\theta).$$

Plugging this in to the expression for the strain, $\varepsilon = \frac{L - L \cos(\theta)}{L \cos(\theta)} = \frac{1 - \cos(\theta)}{\cos(\theta)}$, we get $0.015$ for $\theta = 10^\circ$.

To determine the Young’s modulus we need to know both the stress and the strain. For the stress we first must calculate the tension force in the silk thread. From the $y$-component of Newton’s 2nd law for the spider (see FBD to right), we see that

$$\sum F_y = 2F_T \sin(\theta) - mg = 0 \text{ or } F_T = \frac{mg}{2\sin(\theta)}.$$

The Young’s modulus $Y$ is stress divided by strain or

$$Y = \frac{\sigma}{\varepsilon} = \frac{F_T}{\varepsilon} = \frac{mg}{2\sin(\theta)} \cdot \frac{\cos(\theta)}{1 - \cos(\theta)},$$

which works out to be $190 \text{ MPa}$. This is an order of magnitude smaller than the Young’s modulus for silk we saw in Lecture 16, but it still seems reasonable.

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3. Under pressure (3 pts) Your blood pressure (typically reported as “120 over 80” or something like that) has two parts: the first number is the systolic pressure, which is the arterial pressure during the contraction of the heart, and the second number is the diastolic pressure, which is the arterial pressure when the heart is relaxed.

a) Blood pressures are typically given in torr, or millimeters of mercury (1 torr = 1 mmHg). Using the fact that mercury has a density of $13.6 \text{ g/cm}^3$, derive a conversion factor between torr and pascals, and use it to convert 120 torr into Pa.

In mercury, the pressure difference resulting from a 1 mm change in height is:

$$\Delta P = \rho_{\text{Hg}} gh = (13600 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(0.001 \text{ m}) = 133 \text{ Pa}.$$

So 120 torr = (120 mmHg)(133 Pa/mmHg) = 16.0 kPa.
b) These blood pressures are known as gauge pressures, meaning that they are the amount by which the blood pressure exceeds atmospheric pressure. The first measurement of blood pressure involved inserting a long cannula (needle-tipped tube) into the artery of a horse. The other end of the tube was left open to the atmosphere. If the horse's systolic blood pressure were 150 mmHg, how high would the blood from that artery have risen in the cannula? Assume that blood has the same density as water.

The pressure at the top of the cannula is atmospheric (because the fluid there is open to the atmosphere), and the pressure at the bottom would be the blood pressure, which maxes out (during the systole phase of the heart) at 150 mmHg higher than atmospheric. In addition, we know that the pressure difference between the top and bottom of the column of blood in the cannula is $\Delta P = \rho_{\text{blood}} gh$.

If the cannula were filled with mercury instead of blood, the column of mercury would be 150 mm high; since the density of blood is 13.6 times less than that of mercury, the column will be 13.6 times higher, or $h = \frac{2.04}{13.6} = 0.15 $ m.

c) As we’ll see when we study fluid dynamics, fluids flow from high pressure to low pressure. Using your results from parts a) and b), why must IV bags be placed at least 20 cm above a patient’s arm?

An IV bag must be placed at least 20 cm above a patient’s arm to ensure the fluid in the bag flows into the bloodstream. That is the hydrostatic pressure of the IV fluid must be larger than the blood pressure of the patient. In practice, IV bags are hung at least 1 meter (~ 3 feet) above the patient’s arm to insure a constant flow of the fluid into the patient.

4. Head over heels (3 pts) Blood pressures are measured at the height of the heart. If the maximum (systolic) pressure at the brain drops below zero (gauge pressure), then no blood will reach the brain. This condition can lead to loss of consciousness. If the systolic pressure at the brain exceeds zero but the diastolic pressure does not, blood will reach the brain sporadically instead of continuously. This can cause dizziness, but usually not blackout.

a) What is the minimum pressure (measured at the heart) needed to avoid dizziness? Is this the minimum systolic or diastolic pressure?

Dizziness can be avoided by keeping the diastolic pressure at the head above zero (gauge). My head is about 50 cm above my heart. The minimum diastolic pressure at the heart is therefore

$$P_{\text{heart}} = P_{\text{head}} + \rho g H = 0 + (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(0.50 \text{ m}) = 4900 \text{ Pa} = 36.8 \text{ torr}.$$
b) A giraffe’s head is about 3 meters above its heart. The giraffe’s circulatory system must be specially adapted for its unusual anatomy. What must be the minimum diastolic pressure at the heart of a giraffe? If the systolic:diastolic ratio is the same for giraffes as for humans, what is the systolic pressure of a giraffe at the level of its heart?

For the giraffe,

\[ P_{\text{heart}} = P_{\text{head}} + \rho gH = 0 + (1000 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(3 \text{ m}) = 220 \text{ torr}. \]

If the systolic:diastolic ratio is the same as in humans (about 3:2), then the systolic pressure at the giraffe’s heart will be about \( P_{\text{systolic}} = 330 \text{ torr}. \)

c) Conversely, the blood pressure in your feet is greater than the pressure at your heart. Estimate the maximum (systolic) blood pressure in your feet, if the systolic pressure at your heart is 120 mmHg. This increased pressure can lead to swelling of the feet.

My heart is about 1.17 m above my feet, so the systolic pressure at my feet is:

\[ P_{\text{feet}} = P_{\text{heart}} + \rho gh = 120 \text{ torr} + (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(1.17 \text{ m}) = 206 \text{ torr}. \]

d) Estimate the maximum (systolic) blood pressure in giraffe’s feet, which are 2.5 meters below its heart. Giraffes have very tight elastic skin around their ankles to support this pressure and prevent swelling.

For the giraffe,

\[ P_{\text{feet}} = P_{\text{heart}} + \rho gh = 330 \text{ torr} + (1000 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(2.5 \text{ m}) = 510 \text{ torr}. \]