Physics 239

Lecture 11

Loose Ends:

- c.f. Lodato for Disk LCO
  Disk outburst.

- Alternate of key derivation:

\[ T = \frac{dL}{dt} \]

\[ \rightarrow \text{loss of angular momentum} \]
\[ \text{torque exerted} \]
\[ \text{on outer region} \]

\[ \text{due to accretion} \]
\[ \frac{dM}{dt} = \dot{M} (r^2dr)^2 \Omega (r^2dr) - \dot{M} (r^2dr) \]
\[ \Rightarrow \dot{M} \frac{d}{dr} (r^2dr) \]

So necessary only:

\[ \dot{M} \frac{d}{dr} (r^2dr) = -\frac{d}{dr} (T_{\phi \phi}) \]
From before:

\[
\frac{d}{dn} \left( r^2 \Sigma \right) = -d \left[ \frac{d}{dn} \left( r \sum 2\pi r^3 \Sigma \right) \right]
\]

Integrating:

\[
r^{1/2} \dot{M} = r \sum 3\pi r^{1/2} + c
\]

\[
r \sum r^{1/2} = \frac{\dot{M}}{3\pi} r^{1/2} + c.
\]

Now shear vanishes at some inner radius, \( r_n \) (no torque assumption)

\( r_n \) is object dependent.

\[
r \sum = \frac{\dot{M}}{3\pi} \left[ 1 - \left( \frac{r_n}{r} \right)^{1/2} \right]
\]
\[ D(v) = 4 \sum (v \Delta v / \Delta N)^2 \]

\[ = \frac{3 \rho M \mu^2}{4 \pi \nu^3} \left[ 1 - \left( \frac{\mu}{\nu} \right)^2 \right] \]

etc.

[\text{n.b. BL contributes to luminosity}]

\[ \rightarrow \text{Dynami} \]

\[- \text{ what of viscosity.} \]

\[- \text{ consider collisional viscosity } \]

(c.f. as calculated by Boltzmann Equation and Chapman-Enskog)

\[ \nu = C_S \lambda_{mp}, \quad \lambda_{mp} / \lambda_c \ll 1 \]

\[- \text{ but, disks are } H \rightarrow \Omega_{\text{mag}} \]

(why?)
\[
\frac{\partial f}{\partial \mu} + u \cdot \frac{\partial f}{\partial u} + E \cdot df = CCA \\
f = f_{\text{max}} + df
\]

CCA = \text{0} \quad \rightarrow \text{local Maxwell.}

\[\text{V} \sim \frac{\pi b^3}{1} \quad , \quad \frac{Ze^2}{b} \sim \frac{3}{2} k_b T\]

\[\Rightarrow \text{Vconv feasible.}\]

- Typically,
  \[\text{Re} \sim 2 \times 10^9 \left( \frac{M}{M_0} \right)^{1/2} \frac{R}{10^{10} \text{cm}} \]
  \[ \Rightarrow \Rightarrow \text{t} \]

n.b. \[\text{Re} \sim \frac{V \cdot \text{DV}}{\text{VDV}} - \frac{UL}{R} \]

\[V \sim V_h \quad , \quad L \sim R \quad \text{etc}\]

\[\Rightarrow \text{Disks are enormous Reynolds# systems}\]

\[\Rightarrow \text{collisional viscosity irrelevant}.\]
Mechanism: Turbulence

resulting from instability

- Stability of dish (hydro, MHD)
- Relaxation process

much like pipe flow, etc.

d.e. Law of the wall, due turbulent mixing, explains profile of turbulent pipe flow (c.f. Prandtl Mixing Length).

In practice, find instability

2. Calculate/estimate

\( \nabla_y \cdot \mathbf{f} \rightarrow \text{mixing length} \)
Viscosity model: \( \nabla = \alpha C_0 H \)

Some facts:
- Disks are ubiquitous, so mechanism should be generic.
- \( \nabla \) will be: parameter, spatially dependent, in homogeneous, involve threshold.

And, but
- Disks are remarkably stable.

The classic - Lord Rayleigh

\[ \begin{align*}
\n\end{align*} \]
Interchange

- disk - a set of rings

Does interchange of two rings raise or lower energy?

ΔE: Raise = stable

lower = unstable

-1 moved to indicate perturbation

Consider interchange 1, 2

\[ \Delta E = E_{after} - E_{before} \]

Each ring has angular momentum (~ uniform density)
\[ \Delta E = \left( \frac{L_i^2}{2r_i^2} + \frac{L_2^2}{2r_2^2} \right) - \left( \frac{L_i^2}{2r_i^2} + \frac{L_2^2}{2r_2^2} \right) \]

\[ \uparrow \quad \uparrow \quad \downarrow \quad \downarrow \]

\[ \Theta = \Theta \quad \Theta = \Theta \]

N.B.: Axisymmetric displacement conserves angular momentum of each ring individually.

\[ \Delta E = \frac{L_i^2}{2} \left( \frac{1}{r_i^2} - \frac{1}{r_2^2} \right) + \frac{L_2^2}{2} \left( \frac{1}{r_2^2} - \frac{1}{r_i^2} \right) \]

\[ = \frac{1}{2} \left( L_1^2 - L_2^2 \right) \left( \frac{1}{r_i^2} - \frac{1}{r_2^2} \right) \]

\[ \delta \quad \delta \quad \delta \quad \delta \quad \delta \quad \delta \quad \delta \quad \delta \quad \delta \]

but disk Keplerian

\[ L \sim r^2 \Rightarrow r \sim \sqrt{L} \quad \Rightarrow \text{increases with radius} \]

\[ L_i^2 \geq L_2^2 \quad \text{as:} \]
\( \Delta E > 0 \Rightarrow \) stable.

- Interchange increase energy
- Stable radial stratification

To calculate systematically:

\[
\varepsilon \left( \frac{\partial v_y}{\partial t} + u \cdot \nabla u \right) = -\nabla P \\
\n\n\n\n\]

\[
\rho \partial_+ \tilde{v}_r = -\partial_r \tilde{P} + 2 \Sigma \nu \tilde{v}_r \\
\rho \partial_+ \tilde{v}_z = -\partial_z \tilde{P} \\
\text{and, since axisymmetric perturbations:} \\
\partial_+ r \tilde{v}_r = \tilde{v}_r \partial_r (r^2 \Sigma) \\
\text{adduction of specific angular momentum}
Much akin to Rayleigh-Benard convection with:

\[ \text{Temperature} \to L_0 \]
\[ \nabla T \to a \times (r^2 \Delta) \]

Routine clock \( \to \)

\[
\omega^2 = \frac{k_r^2}{k^2} \Phi
\]

\[
\left(\frac{k^2}{k_r^2} + \frac{k_r^2}{k_r^2} \right) \Phi = \frac{2L}{r} \frac{\partial}{\partial r} (r^2 \Delta)
\]

Rayleigh discriminant

\( \Phi > 0 \) for stable stratification

- Uniform rotation
  \[ \Phi = 4L^2 \]
  Chemical wave.

- Solberg-Holm and waves

- Also, no inflection point in \( V_0(r) \)
  (Rayleigh inflection point theorem)
Other candidates:
- convection - planet forming disks
- MRI

TBC.

Instability $\Rightarrow$ relaxation

Relaxation $\Rightarrow$ constrained minimum energy state?

What is it? How?

Lynden-Bell and Pringle

- MNRAS 1974, posted
- a classic - need it!

"Discs have played a large part in astrophysical thought since it was discovered that the solar system is flat."
Now, would expect:

- dissipation leads to rigid rotation (no shear stress)

- minimum energy state is uniform rotation. → Prove etc!

but of course segregation and accretion happens.

How understand and reconcile?

Hint: Interplay of mass and angular momentum transport!

Thm: For given density distribution and total angular momentum, the motion of least energy is uniform rotation.
- $E_{\text{extern.}}, E_{\text{grav}}$ fixed

- \[ T = \frac{1}{2} \int \rho V^2 \, d^3x = \frac{1}{2} \int V^2 \, dm \]

- \[ L = \int r V \, dm \]

- \[ I = \int r^2 \, dm \]

- \[ \text{Fixed} \]

- $dm = \rho d^3x$

- \[ \text{Moment of inertia} \]

- Then:

- \[ \int V \rho^2 \, dm \cdot I \, dm = \int V \rho^2 \, dm \int r^2 \, dm \]

- \[ \geq \left( \int r V \rho \, dm \right)^2 \quad \text{by Schwarz Inequality} \]

- \[ \frac{K}{\rho} \]

- \[ \int V \rho^2 \, dm \cdot I \geq L^2 \]

- $1 \rho^2$
\[ \int \nu^2 \, dm \geq \frac{L^2}{I} \]

Now, exactly if \( \nu = \frac{L}{\sqrt[3]{I}} \)

i.e.

\[ \int \nu^2 \, dm = L^2 \cdot \int \nu^2 \, dm = \frac{L^2}{I} \]

Solid body rotation is minimum energy state.

But how account for accretion happening?

N.B.: \( T_{JJ} \sim \Sigma v r \frac{d\Omega}{dr} \)

Solid body \( \Rightarrow \Omega' = 0 \) (zero spin)

But how reconcile?

\( \Rightarrow \) re-visit two particle (ring) argument!
Sodium:

- Lower energy
- Conserve total angular momentum and conserve total mass

N.B.: Important:

Rayleigh: conserves angular momentum of each particle

L-B & P: conserves total angular momentum and mass of pair

Pair can interact so long as total angular momentum conserved.

Why 2? Account for mass and angular momentum.

Interaction anticipates MRI.
$$\frac{L}{m} = h \Rightarrow \text{specific angular momentum}$$

$$\frac{E}{m} = \xi \Rightarrow \text{specific energy}.$$  

\[
\xi = \frac{1}{2} (U_n^2 + U_z^2) + \frac{h^2}{2r^2} - \Psi(r, z) \\
\text{maximal on } z = 0.
\]

So, minimal $$\xi$$ for given $$h$$ (express $$\xi$$ in terms of $$h$$)

\[
\begin{cases}
U_n = U_z = 0 : \\ \\
\text{eq. of } r \\ \\
e \xi = 0
\end{cases}
\]

\[
\text{So } \xi(h) \gtrless R_h \Rightarrow \text{radius of minimum energy circular orbit}
\]

\[
\Rightarrow E(h) = \frac{1}{2} \frac{h^2}{2r^2} - \Psi(h, 0)
\]
\[ 
E(h) = \frac{v^2}{2} - \Psi \\
\therefore v = \frac{h}{r_n} 
\]

\( \Rightarrow \) minimum energy.

Then:

\[ 
\frac{dE}{dh} = \frac{dE}{dh} = \frac{d\Psi}{dh} = \frac{1}{r_n^2} = \Omega \\
\]

NB: \( E(h) \) already stationary with variations

cie \( \frac{dE}{dh} = \frac{d\Psi}{dh} + \frac{d\Psi}{dr} \frac{dr}{dh} \)

Now:

First (I)

\( \Rightarrow \) Minimize energy of 2 particles,

Keep \( r_1 + r_2 = \text{constant} \).

Angular momentum

\( \Rightarrow \) Minimize energy of each at \( L \) const,

\( \Rightarrow \) 2 circles

- tower \& minimize via exchange?
Each particle: \( E_i = m_i \epsilon_i \)

\[ E = m_1 \epsilon_1 + m_2 \epsilon_2 \]

\[ H = m_1 h_1 + m_2 h_2 \]

\[ dH = 0 \quad m_1 dh_1 + m_2 dh_2 = 0 \]

\[ dE = m_1 dh_1 \epsilon'(h_1) + m_2 dh_2 \epsilon'(h_2) \]

\[ = m_1 dh_1 [ \epsilon'(h) - \epsilon'(h_2) ] \]

\[ \Sigma_1 > \Sigma_2 \]

\( \text{If} \ dE < 0 \rightarrow \text{Relax} \)

\( \text{dh}_1 < 0 \text{ so } \text{dh}_2 > 0 \)
Orbit 2, of lower angular velocity, gains angular momentum while orbit 1, of higher angular velocity, lowers/loses angular momentum.

Minimization (relaxation) of energy by exchange of angular momentum to orbit of lower \( S \) \( \Rightarrow \) outward transfer angular momentum.

Energy lowered if angular momentum flows/transported outward.
Now, consider that total mass fixed ± orbits exchange mass, too, as well as angular momentum.

\[-dE = d \left[ m_1 E(ch_1) + m_2 E(ch_2) \right] \]

\[= dm_1 = 0 \quad dh_1 = dm_2 \]

\[-dH = 0 = dh_1 + dh_2 \]

\[d(m_1 h_1) + d(m_2 h_2) = 0 \]

Now,

\[dE = dm_1 E(ch_1) + m_1 E'(ch_1) dh_1 \]

\[+ dm_2 E(ch_2) + m_2 E'(ch_2) dh_2 \]

\[= dm_1 \left[ E(ch_1) - h_1 E'(ch_1) \right] + d(m_1 h_1) E'(ch_1) \]

\[+ dm_2 \left[ E(ch_2) - h_2 E'(ch_2) \right] + d(m_2 h_2) E'(ch_2) \]
\[ \text{a) } E = \text{d}m \left\{ \left[ E(h_{\lambda}) - h_{\lambda} \Omega_{\lambda} \right] \right\} \]

\[ \text{b) } E = \text{d} \Omega_{\lambda} \left( \Omega_{\lambda} - \Omega_{\rho} \right) \]

\[ + \text{d} m \left\{ E(h_{\lambda}) - h_{\lambda} \Omega_{\lambda} \right\} - \left\{ E(h_{\rho}) - h_{\rho} \Omega_{\rho} \right\} \]

1. \( \Omega_{\lambda} > \Omega_{\rho} \)
   \( \text{d} \Omega_{\lambda} < 0 \)

2. Need profile \( E = h \Omega \)

\[ \frac{d}{dr} (E - h \Omega) = \frac{d}{dr} \left( -\frac{1}{2} \frac{V^2}{r} - \Phi \right) \]

\[ = -V \frac{dV}{dr} + \frac{V^2}{r} \]

\( \Phi \) is const \( \text{h} = 1 \)

\[ = -V \left( \frac{dV}{dr} - \frac{V}{r} \right) \]
\[ = -rV \frac{d}{dr} \left( \frac{V}{r} \right) > 0 \]

\[ \frac{d}{dr} (E - h \ell) > 0. \]

\[ dE = d\ell \{ (L_1 - L_2) \} > 0 \]

\[ + \Delta m_1 \{ A (E - h \ell) \} < 0 \]

\[ dE < 0 \Rightarrow d\ell < 0 \]

\[ \Delta m_1 > 0 \]

\[ dE < 0 \text{ for:} \]

\[ \Rightarrow (\ell) \text{ gains angular momentum} \]

\[ \Rightarrow \text{angular momentum coupled outward}. \]
\( \rightarrow \) Germs mass

\[ \text{i.e. mass accretes}. \]

\( \rightarrow \) Accretion - with angular momentum transport outward - is an energy-minimizing relaxation process.

\( \rightarrow \) Deviation / Departure from solid body comes from mass accretion.

\( \rightarrow \) End state

1 particle

\[ \begin{align*}
\text{end state} & \quad \text{1 particle} \\
\bullet & \quad \text{all}
\end{align*} \]

Full \& angular momentum

\( \rightarrow \) Final minimum energy state is

- full accretion

- 1 particle at \( \infty \) to carry angular momentum.
Quite a contrast to solid body $\frac{1}{2}$.

TBC.