Lecture 9

ORTHOGONALITY AND GRAM-SCHMIDT

Last time:

MATRIX RANK

- (column rank) \( \max \# \) of LI columns
- (row rank) \( \max \# \) of LI rows
- (Gaussian elim.) \# of pivots
- (factorization) \( \tilde{A} = \tilde{B} \tilde{C} \) smallest dimension \( k \)
ORTHOGONALITY:

- Two vectors are orthogonal if the angle between them is $90^\circ$ (or $\pi/2$) (plane geometry).

(also "perpendicular")

DEF: Vectors $v$ and $w$ are orthogonal if $v \cdot w = 0$ (denoted as $v \perp w$).

Recall: $v \cdot w = v^T w = \sum v_i w_i$.

$w \cdot v = w^T v = \sum w_i v_i$. 
WHY ORTHOGONALITY?

\[ \| v - w \|^2 = \| v \|^2 + \| w \|^2 \]

or

\[ \| v + w \|^2 = \| v \|^2 + \| w \|^2 \]

**Proof:**

\[ \| v + w \|^2 = (v+w) \cdot (v+w) \]

\[ = v \cdot v + v \cdot w + w \cdot v + w \cdot w = \| v \|^2 + \| w \|^2 \]
USEFUL!

**Norm of Sums:**

E.g., if \( \{ v_1, \ldots, v_k \} \) are **pairwise orthogonal**

Then

\[
\left\| \sum \lambda_i v_i \right\|^2 = \sum \lambda_i^2 \left\| v_i \right\|^2
\]

**Linear Independence:** \( \{ v_1, \ldots, v_k \} \)

**Pairwise Orthogonal Vectors are LI:**

Why?

(\( \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k \)) = 0

\( \lambda_1 (v_1 \cdot v_1) + \lambda_2 (v_2 \cdot v_2) + \cdots + \lambda_k (v_k \cdot v_k) = 0 \)

\( \lambda_1 = 0 \)
WHY SHOULD WE CARE?

ENABLES US TO EXTEND GEOMETRIC IDEAS TO MANY OTHER OBJECTS (AS LONG AS THEY FORM A VECTOR SPACE)

Examples:

- POLYNOMIALS
  \[ p(x) = x^m + \ldots + a_0 \]
- FUNCTIONS
- SIGNALS (AUDIO, VIDEO, WIRELESS)
- RANDOM VARIABLES
  \[ \text{Gaussian } N(0, 1) \]

(And many more!)
ORTHOGONALITY OF SUBSPACES

Two subspaces $V, W \subseteq \mathbb{R}^n$ are orthogonal if $V \cap W = \{0\}$ for every $\{v \in V \mid w \in W\}$.

(Not the only possibilities!)
Q: If $V$ and $W$ are orthogonal, what can we say about their dimensions?

E.g., can we have two orthogonal planes in $\mathbb{R}^3$?

If $V, W$ are orthogonal, can they intersect?

$\Rightarrow \text{NO!}$

$\exists \ p \in V, \ p \in W \Rightarrow p = 0$

$\dim(V) + \dim(W) \leq n$
ORTHOGONAL COMPLEMENT

**Def:** Given $V \subseteq \mathbb{R}^n$, its orthogonal complement $V^\perp$ is defined as

$$V^\perp = \{ w \in \mathbb{R}^n : w \cdot v = 0 \quad \forall v \in V \}$$

"vectors that are orthogonal to all $v \in V$"

Q: Is $V^\perp$ a subspace?

- closed under addition ✓
- "" under multiplication by scalars.

$w_1 \in V^\perp \implies w_1 \cdot v = 0 \quad \forall v \in V$

$w_2 \in V^\perp \implies w_2 \cdot v = 0 \quad \forall v \in V$

$w_1 + w_2 \in V^\perp \implies (w_1 + w_2) \cdot v = 0 \quad \forall v \in V$
Q: What can we say about \( \dim (V^\perp) \)?

\[
\dim (V) + \dim (V^\perp) = n
\]

Q: What can we say about \((V^\perp)^\perp\)?
Def: If \( V, W \) are orthogonal complements of each other (i.e., \( V = W^\perp, \; w = V^\perp \)) they form an orthogonal decomposition.

Then, every vector \( x \) in the space can be written uniquely as

\[
x = v + w
\]

where

\[
\begin{align*}
v & \in V \\
w & \in W
\end{align*}
\]

\( v \cdot w = 0 \)

Q: Where do these decompositions come from?
Hold on...

Even before:

Q: Where do orthogonal vectors come from??

We can make them!

**Gram-Schmidt**: a procedure (algorithm) to orthogonalize a set of LI vectors
\{v_1, \ldots, v_n\}

\textbf{Algorithm}

\{w_1, \ldots, w_n\}

\textbf{orthonormal vectors}

\|w_i\| = 1

w_i \cdot w_j = 0 \quad i \neq j

\underline{Projection:} \quad \text{Given } w, \text{ with } \|w\| = 1.

The projection of \(v\) onto \(w\) is

\[ \text{proj}_w v = (\mathbf{v} \cdot \mathbf{w}) \mathbf{w} \]

Define:

\[ \begin{cases} s = \text{proj}_w v \\ t = v - \text{proj}_w v \end{cases} \]

\underline{Claim:} \quad v = s + t, \quad s \cdot t = 0

\[ s \cdot t = (\alpha w) \cdot (v - \alpha w) = \alpha (\mathbf{v} \cdot \mathbf{w}) - \alpha^2 \mathbf{w} \cdot \mathbf{w} = 0. \]
Input: \{v_1, \ldots, v_n\} LI vectors

Idea: At every step, subtract the projection onto all previously found vectors.

\[ v_1: \quad w_1 = \text{normalize} (v_1) \quad w_i = \frac{v_i}{\|v_i\|} \]

\[ v_2: \quad w_2 = \text{normalize} (v_2 - \text{proj}_{w_1} v_2) \]

\[ v_3: \quad w_3 = \text{normalize} (v_3 - \text{proj}_{w_1} v_3 - \text{proj}_{w_2} v_3) \]

\vdots

\[ v_n: \quad w_n = \text{normalize} (v_n - \sum_{k=1}^{n-1} \text{proj}_{w_k} v_n) \]

Claim:

- \{w_1, \ldots, w_n\} are orthonormal

- \text{Span} \{w_i\} = \text{Span} \{v_i\}

i.e., an "orthonormal basis"
A natural source of orthogonal decompositions: MATRICES!

Given \( A \in \mathbb{R}^{m \times n} \)

\( N(A) \subseteq \mathbb{R}^n \) and \( C(A) \subseteq \mathbb{R}^m \)

Great. But, we need more...

We also have \( A^T E \in \mathbb{R}^{n \times m} \)!
$A^T \in \mathbb{R}^{n \times m}$

where is $N(A^T)$?

$C(A^T) \subseteq \mathbb{R}^n$

$N(A^T) \subseteq \mathbb{R}^m$

Hmm...

$\mathbb{R}^n$

$\mathbb{R}^m$

$N(A)$

$C(A^T)$

$C(A)$

$N(A^T)$

Why?
They are orthogonal:

Take $v \in N(A)$, $w \in C(A^T)$.

Then $w \cdot v = w^T v = (A^T z)^T v = z^T A v = 0$

$w = A^T z$

(same for other).

Why do they fill the space?

What about dimensions?

If $r = \text{rank}(A) = \text{rank}(A^T)$

then

$r = \text{rank-nullity}$

$\dim C(A^T) = r$ \hspace{1cm} $\dim C(A) = r$

$\dim N(A) = n-r$ \hspace{1cm} $\dim N(A^T) = m-r$

$\mathbb{R}^n$ \hspace{1cm} $\mathbb{R}^m$
Thus, any vector \( \mathbf{v} \in \mathbb{R}^n \) can be decomposed uniquely as

\[
\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2
\]

with \( \mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \), \( \mathbf{A}\mathbf{v}_1 = 0 \) \( \mathbf{v}_2 = \mathbf{A}^T \mathbf{w} \)

Will allow us to understand the structure of matrices (and linear maps) in a simpler way.
THE BIG PICTURE

\[ \mathbb{R}^n \]

\[ C(A^T) \]

\[ N(A) \]

\[ C(A) \]

\[ N(A^T) \]

\[ A \]

\[ A^T \]

\[ \dim \mathbb{R}^r \]

\[ \dim n - r \]

\[ \dim M - r \]

More, next time!