CS30 (Discrete Math in CS), Summer 2021 : Ungraded Practice Problems 1

Due: Not for Submission
Topics: Sets, Function, Logic

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1 Sets

Problem 1.
The following sets are described in set builder notation. Describe each of them in roster notation, instead.

a. $\{z^2 \mid z \in \mathbb{Z} \text{ and } 6 < z^3 < 160\}$.

b. $\{A \mid A \subseteq \{a, c, e\} \text{ and } |A| \neq 2\}$

c. $\{r \mid r \in \mathbb{R} \text{ and } r = r^2\}$

d. $\{n \mid n \in \mathbb{N} \text{ and } n > n^2\}$

e. $\{p \in \mathbb{Z} \mid 0 < p < 50 \text{ and } p \text{ does not have 3 as a digit }\}$

f. $\{x \mid x \text{ is a letter in the word } \textit{accommodate}\}$

g. $\{S \subseteq \{2, 4, 6, 8\} \mid S \cap \{2, 4\} \neq \emptyset \text{ and } |S| \text{ is even}\}$

Problem 2 (Set Operations)**.**
Let A, B, C be sets. Express the following sets using set operations on A, B, and C. For example, if I ask you to express "the set of all elements that are common to A and B,"' your answer should be "$A \cap B$," and not "$\{x : (x \in A) \text{ and } (x \in B)\}$."

a. Set of all elements that are in at least two of the three sets A, B, C.

b. Set of all elements that are in *exactly* one set A or B or C.

Problem 3 (Set builder Notation)**.**
If S, T are arbitrary sets of integers, express the following sets as elegantly as possible using the set builder notation.

a. The set of all pairs where the first element of the pair is an element of S and the second element of the pair is a subset of S that does not contain the first element of the pair.

b. The set of all subsets of S that are disjoint from T.

c. The set of all numbers that can be expressed as the difference of two numbers in S.

Problem 4 (Applying Baby Inclusion-Exclusion)**.**
In the city of Klow, 80% of the population speak Syldavian while 70% of the people speak Bordurian. Furthermore, 60% of the population speak both languages. If there are 10 residents who speak neither language, what is the population of Klow?

2 Functions

Problem 5 (Proving something is injective).

Let $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ be the set of non-negative real numbers. Let $(0, 1] := \{r \in \mathbb{R} : 0 < r \leq 1\}$. Prove that the following function $f : \mathbb{R}_+ \to (0, 1]$ is valid and injective.

$$f(x) = \frac{1}{1 + x}$$

Problem 6 (Proving something is surjective).

Let $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ be the set of non-negative real numbers. Let $(0, 1] := \{r \in \mathbb{R} : 0 < r \leq 1\}$. Prove that the following function $f : \mathbb{R}_+ \to (0, 1]$ is surjective.

$$f(x) = \frac{1}{1 + x}$$

Problem 7 (Composition).

As you know, given any two functions $g : A \to B$ and $f : B \to C$, the composition of f and g, denoted $g \circ f$, is the function from A to C given by $(g \circ f)(x) = g(f(x))$.

If g and f are functions from $\mathbb{R}$ to $\mathbb{R}$ given by $g(x) = 2x^2 + 5$ and $f(x) = 3x^2 - 1$,

a. What is $(g \circ f)(x)$?

b. What is $(f \circ g)(x)$?

c. What is $(f \circ f)(x)$?

d. What is $(g \circ g)(x)$?

e. What are the values of $(g \circ f)(2)$, $(f \circ g)(2)$, $(f \circ f)(2)$, and $(g \circ g)(2)$?

3 Logic

Problem 8.

Rewrite each of the following statements in English in the form $p \Rightarrow q$. For example, the statement "I catch cold if I eat ice cream" should be rewritten "I eat ice cream $\Rightarrow$ I catch cold."

a. Winds from the south imply a strong thaw.

b. Willy gets caught whenever he cheats.

c. You can access the website only if you pay the subscription fee.

Problem 9. ☕

Let S be the statement $p \Rightarrow q$ for two propositions p and q. Define the *converse*, *inverse*, and *contrapositive* of the statement S to be the statements represented by $q \Rightarrow p$, $\neg p \Rightarrow \neg q$, and $\neg q \Rightarrow \neg p$, respectively. Now consider the following sentences and do what is asked.

a. *I open my umbrella whenever it rains.*

   Rewrite the above sentence in the form $p \Rightarrow q$. Then write a natural sounding English sentence that represents its inverse.

b. *I miss class only if I am unwell.*

   Rewrite the above sentence in the form $p \Rightarrow q$. Then write a natural sounding English sentence that represents its contrapositive.

c. *You can't invent unless you are curious and knowledgeable.*

   Rewrite the above sentence in the form $p \Rightarrow q$, using the symbol $\neg$ wherever necessary. Then write a natural sounding English sentence that represents its converse.

Problem 10. ☕☕

Without using truth tables, and using the important equivalences done in class and given in the lecture notes, prove the following logical equivalences.

a. $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

b. $\Big((p \Rightarrow \neg p) \Rightarrow ((q \Rightarrow (p \Rightarrow p)) \Rightarrow p)\Big) \equiv p$

Problem 11 (Fun with Predicate Logic)**.** ☕☕

Let $P[1 \ldots n, 1 \ldots m]$ be a 2-dimensional array of the pixels of a black-and-white image: for every x and y, the value of $P[x, y] = 0$ if and only if the (x, y)th pixel is black, and $P[x, y] = 1$ if it is white. Translate the following statements into predicate logic.

a. Every pixel in the image is black.

b. There is at least one white pixel.

c. Every row has at least one white pixel.

d. There is no column with two consecutive white squares.