1 Recap

1.1 Linear Independence

A collection of vectors \( \{v_1, \ldots, v_n\} \) is \textit{linearly independent} (LI) if any linear combination that results in a zero vector must be trivial. In other words,

\[
\sum_{i=1}^{n} \lambda_i v_i = 0 \quad \implies \quad \lambda_i = 0 \text{ for all } i = 1, \ldots, n.
\]

We also say that a collection of vectors \( \{v_1, \ldots, v_n\} \) is \textit{linearly dependent} if it is not LI. In other words, there exists scalar multipliers \( \lambda_1, \ldots, \lambda_n \) where at least one of them is non-zero, such that

\[
\sum_{i=1}^{n} \lambda_i v_i = 0.
\]

\textbf{Key Fact:} There can be at most \( n \) linearly independent vectors in \( \mathbb{R}^n \).

1.2 Generators

Let \( S \) be a subspace. We say that \( \{v_1, \ldots, v_k\} \subset S \) are \textit{generators} of \( S \) if every vector \( v \in S \) is a linear combination of \( \{v_1, \ldots, v_k\} \). In other words, \( v = \lambda_1 v_1 + \cdots + \lambda_k v_k \) for some scalars \( \lambda_1, \ldots, \lambda_k \).

We can also write \( S = \langle v_1, \ldots, v_k \rangle \) or \( S = \text{Span}(v_1, \ldots, v_k) \).

1.3 Two Descriptions of Subspaces

Two useful descriptions of subspaces include:

1. Equations. We can describe a subspace \( S \) as a set of vectors satisfying certain linear relationships between their entries.

2. Generators. We can describe a subspace \( S \) as the span of a set of vectors.

For instance, the subspace \( S \) of 3-dimensional vectors whose third entry is the sum of the first and second entries can be expressed as either:

\[
S = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} : v_1 + v_2 - v_3 = 0 \right\} \quad \text{or} \quad S = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}
\]

Depending on the task, one description may be more convenient than the other. We can use Gaussian elimination to go from one description to the other.
1.4 Basis

We say \( \{v_1, \ldots, v_k\} \) is a basis of a subspace \( S \) if they generate \( S \) and are LI.

In general, a subspace has infinitely many different bases. However, they all must have the same cardinality (i.e., the number of vectors in the basis) – this is called the dimension of the subspace.

2 Exercises

1. Note that the notions of linear independence and linear dependence are not quite symmetric. In particular:

   (a) Show that if \( \{v_1, \ldots, v_k\} \) are LI, then any subset of the vectors is also LI.

   (b) Does a similar statement hold for linear dependence? Prove this, or give a counterexample.

2. Identify if the following sets of vectors are linearly independent or not.

   (a) \( A = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \).

   (b) \( B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \).

   (c) \( C = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \).

   (d) \( D = \begin{bmatrix} 0 & 3 & -4 & 9 & 100 \\ -1 & -2 & 0 & -12 & 10 \\ 3 & 11 & 0 & 6 & 1 \end{bmatrix} \).

3. Let \( A = \begin{bmatrix} -3 & 1 & 0 & 5 \\ -2 & 2 & -2 & 1 \\ 1 & -3 & 4 & 3 \end{bmatrix} \). Answer the following questions.

   (a) Are the columns of \( A \) linearly independent?

   (b) Find a set of generators for \( N(A) \), the nullspace of \( A \).

   (c) Find a basis of the column space \( C(A) \), aka the span of columns.

   You can do it either by inspection, or algorithmically, using Gauss-Jordan elimination.

4. Consider the two subspaces
\[
U = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}\right\}
\]

\[
V = \left\{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} : v_1 + 2v_2 - 4v_3 = 0\right\}
\]

(a) Write a description of \( U \) as a set of vectors that satisfy linear relationships.
(b) Write a description of \( V \) as the span of a set of generators.
(c) Compute the dimension and a basis of \( U \) and \( V \).
(d) Compute the dimension and a basis of \( U \cap V \).