Math 108 - Geometric Combinatorics Nadia Lafrenière

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Posets

A <u>partially ordered set</u> (<u>poset</u>) P is a set together with a binary relation $\leq$ that satisfies

(i) For all $t \in P$, $t \leq t$ Reflexivity

(ii) If $s \leq t$ and $t \leq s$, then $s = t$ Antisymmetry

(iii) If $s \leq t$ and $t \leq u$, then $s \leq u$. Transitivity.

(The symbols $<$, $>$ and $\geq$ are also defined.)

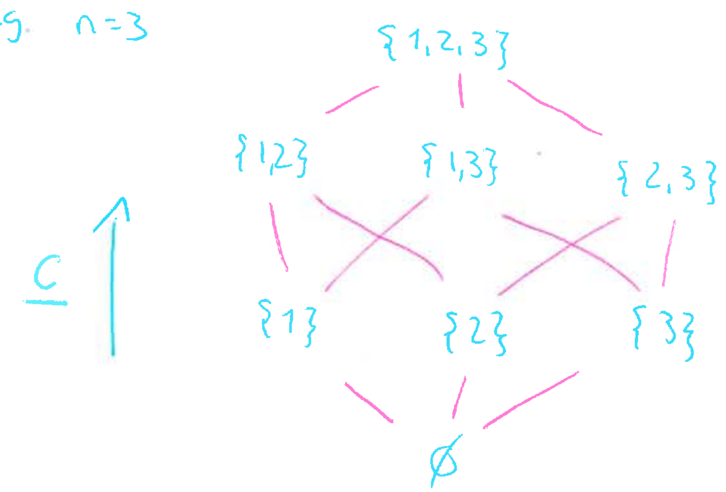
Two elements s and t are <u>comparable</u> if $s \leq t$ or $t \leq s$, and <u>incomparable</u> otherwise.

Examples

(i) $[n]$ along with the usual order on the natural (or real) numbers is a poset. Here, it has the special property that every pair of elements are comparable.

(ii) The subsets of $[n]$, with the inclusion.

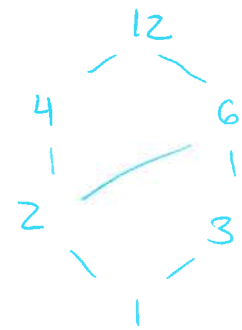
E.g. $n=3$



subsets

(iii) The poset D_n is the poset of all divisors of n, with the divisibility relation

E.g. $n = 12$



divisors lattice

(iv) Integer partitions, with $\lambda \leq \mu$ if they are partitions of the same number n, and if $\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$, $\forall j \in [n]$

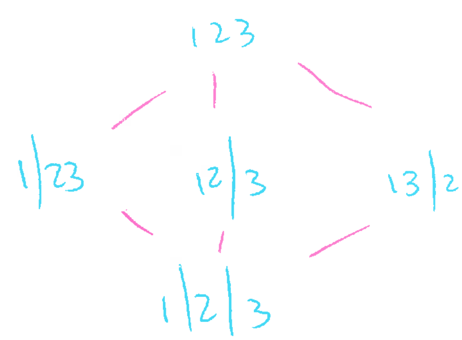
E.g. $(3,1,1) \geq (2,2,1)$

dominance order for partitions

$(3,3)$ and $(4,1,1)$ are incomparable.

(v) Set partitions, with refinement, i.e. $\pi \leq \sigma$ if every block of π is contained in a block of σ.
The opposite of the refinement is the coarsening.

E.g. partitions of 3



refinement for set partitions

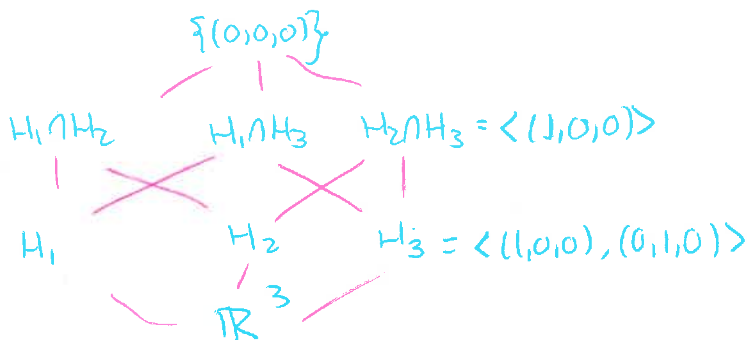
Definition

Two posets P and Q are isomorphic, denoted $P \cong Q$, if there is an order-preserving bijection ψ; that is

$$s \leq t \text{ in } P \iff \psi(s) \leq \psi(t) \text{ in } Q.$$

For example, the subsets in (ii) form a poset that is isomorphic to the boolean lattice defined as following: Let $\{H_1, H_2, \ldots, H_n\}$ be the hyperplanes normal to the canonical vectors in $\mathbb{R}^n$, and let the items in the poset be their intersections. The order is reverse inclusion.

E.g. $n=3$



Boolean lattice

Definition

If s and t are in P, $s<t$, and there is no u in P such that $s<u<t$, then t covers s, denoted $s \lessdot t$.

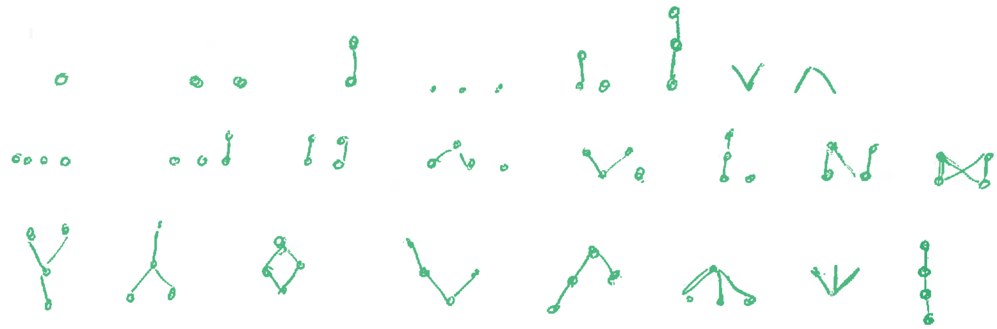
The Hasse diagram of P is a graph whose vertices are items in the set and whose edges are cover relations. We draw s above t is $s \geq t$.

Example: All the pictures in the last 3 pages.

Proposition

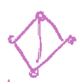
Two posets are isomorphic iff they have the same Hasse diagram.

List of all posets with at most four elements



n	# posets
1	1
2	2
3	5
4	16
5	63
6	318
7	2045
8	16999
n	$\sim 2e^{n^2/4}$

Problem: How many non-isomorphic posets with n elements are there? (open)

Remark: where are  and  ?

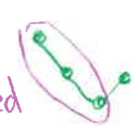
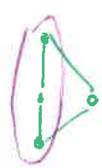
Hint: They are not Hasse diagrams...

We say that P has a $\hat{0}$ if there is an element $\hat{0}$ such that $t \geq \hat{0}$ for all $t \in P$; $\hat{0}$ is the minimum. P has a $\hat{1}$ (maximum) if there is an element $\hat{1}$ such that $t \leq \hat{1}$ for all $t \in P$.

Definition

A chain is a subset of a poset that is totally ordered.

- It is maximal if it is not included in another chain.
- It is saturated if there exists no element in the complement that is included between two elements of the chain and that adding that element would make it a chain.

maximal & saturated   maximal and saturated  saturated, but not maximal

Maximal $\Rightarrow$ saturated
$\not\Leftarrow$

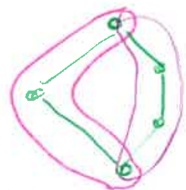
- If every maximal chain has length n, the poset is said to be graded of rank n. The rank is the distance to a minimal element.

Example

All posets from before are graded.

The Tamari lattice is not graded, since its Hasse diagram is (for the lattice of order 3) 

maximal of length 2



maximal of length 3

(the Tamari lattice is describing ways of grouping sequences in pairs using parentheses.)

a(b(cd))

a((bc)d)

(a(bc))d

(ab)(cd)

((ab)c)d

- An antichain is a subset of elements that are all pairwise incomparable.

  Objects of the same rank always form an antichain, but this is not a requirement.

- Two elements s and t have a
  - least upper bound u, called the join, if $u > s, u > t$, and if $v > s$ and $v > t$, then $v > u$. We denote it $u = s \vee t$. ("s join t").
  - greatest lower bound w, called the meet, if $w < s, w < t$, and if $v < s$ and $v < t$, then $v < w$: $w = s \wedge t$ ("s meet t").

- A lattice is a poset in which each pair has a meet and a join.

Exercise: Which posets of 4 elements are lattice?

Reference: Richard P. STANLEY. Enumerative Combinatorics,
Vol. 1.
Sections 3.1 and 3.3.