Lecture 10b: Microscopic Model of Diffusion

Learning objectives: After this lecture, you will be able to:

1. Explain the quantitative features of a random walk and how we can use it to model Brownian motion and diffusion
2. Show that the mean square displacement is linear in time and directly proportional to the diffusion coefficient
3. Explain that the diffusion coefficient tells us how long it takes a molecule (or particle) in solution to move (on average) a given distance.
4. Understand where and how small living cells rely on diffusion for transport.
5. Understand how the Einstein-Stokes equation can be used to estimate the diffusion constant.

Pre-reading: Again, most of this lecture will be new to you, but we will rely heavily on your knowledge of average thermal energy $k_B T$ and Brownian motion. Please review activity 1 in Lecture 10a.

Activity 1: Why does Brownian motion look the way it does?

1. Example: the protein Lysozyme

   \[ M = \frac{14 \text{kDa}}{6.022 \times 10^{23}} \]

   (a) The protein Lysozyme has a molecular weight of 14 kDa
   (mass of one mole = 14 kg).
   What is its root-mean-square velocity when it is dissolved in water? Assume $T = 20^\circ C$

   \[ \left\langle \nu^2 \right\rangle = \frac{1}{2} m \left\langle \nu^2 \right\rangle = \frac{3}{2} k_B T \]

   \[ \nu_{rms} = \sqrt{\frac{3 k_B T}{m}} = \frac{2.3 \text{m/s}}{50 \text{mph}} \]

   (b) The velocity we obtained for the protein seems really large. If we were to look at it in a microscope, would we see it zipping through the fluid at $\nu_{rms}$? Explain
   (Recall youtube videos of Brownian motion).

2. Qualitatively, explain how the Brownian motion of particles (e.g. pollen) changes if:

   (a) the temperature of the fluid is increased

   \[ T \uparrow, \quad \text{speed of pollen} \uparrow \]

   (b) the mass of the particle is decreased/increased (keeping the volume constant)

   \[ \text{mass of pollen} \downarrow, \quad \text{H}_2\text{O molecules will have negligible effect} \]

   Brownian motion will "decrease"
Summary: Kinetic theory and Boltzmann distribution

(1) Everything in a fluid exhibits fluctuating, random motions...
...hence things inside living cells also exhibit fluctuating, random motions

(2) The average kinetic energy of anything in a gas/liquid is about $k_B T$.

\[
In 3-D, with 3 degrees of freedom (x-y-z translation) we have \langle KE \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T
\]

This gives $v_{\text{rms}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3 k_B T}{m}}$

(3) Molecules and particles in a gas/liquid continuously exchange energy through collisions. The Boltzmann distribution tells us the probability that a molecule or particle has energy $U$ given the average kinetic energy $k_B T$.

The Boltzmann distribution is:

\[
\frac{\text{prob(state 2)}}{\text{prob(state 1)}} = e^{-\frac{\Delta U}{k_B T}} = e^{-\frac{U_2 - U_1}{k_B T}}
\]

Today:

- Brownian motion can be modeled as a random walk, allowing us to make quantitative predictions for the mean square displacement

- The mean square displacement is linear in time and directly proportional to the diffusion coefficient

- The diffusion coefficient tells us how long it takes a molecule or particle in solution to move (on average) a given distance. Diffusion is an efficient way to transport molecules in small biological organisms, but not eukaryotic cells.

- The diffusion coefficient is a function of particle size and fluid properties. Einstein developed a model based on four relations:
  * Kinetic theory
  * Random walk diffusion coefficient
  * Random walk friction coefficient
  * Stokes' equation for drag
Activity 1A: Intro to Random Walks

Concept: Small particles in fluids look like they’re doing a “random walk”

Simple model: 1-D random walk
- A particle steps a displacement $\pm \delta$ every $\tau$ seconds by the flip of a coin.
- It doesn’t remember any of its previous steps
- After $n$ steps, what is the average distance away from the origin? $\langle x \rangle = 0$

- What is the average square distance after $n$ steps? $\langle x^2 \rangle = n\delta^2$

(We are only looking at the pollen particle, not the water molecules!)

Einstein’s model of Brownian motion is based on the concept of random walks.

The movements we see are the results of statistical fluctuations. We don’t see individual kicks, but we can see the sum of them (which itself is a random variable)
Activity 1A: Random Walk (acting like particles)

How far could you expect to go by taking steps in random directions?

You are about to do a 1-D random walk. Start in the aisle. Every 5 seconds ($\tau = 5$ seconds) flip a coin. For each flip, if it lands heads then move to your right; if it lands tails then move to your left. Here $\delta$ is the length of 1 step.

1a. After your first flip, how far are you from your start point? Count number of steps to the left as negative and number of steps to the right as positive; for example, “2 steps to the left of the starting point” is -2 and “3 steps to the right of the starting point” is 3.

Number of steps from starting point: ______ (your answer should be 1 or -1)

2a. After your second flip, how far are you from your start point?

Number of steps from starting point: ______ (your answer should be -2, 0 or 2)

3a. After your third flip, how far are you from your start point?

Number of steps from starting point: ______ (your answer should be -3, -1, 0 or 3)

Let’s flip the coin (and step) several more times. Watch how everyone spreads out.

Put your answers from the above into Canvas.

1b. After 1 step: How many people were -1 step from the starting point? 75 (see Canvas results)

How many people were +1 step from the starting point? 87 (see Canvas results)

Calculate the average distance ($x$) from the starting point (in units of steps; 1 step = $\delta$).

$$\frac{(75(-18) + 87(+18))}{162} = 0.78 \approx 0.8$$

Calculate the average ($x^2$) from the starting point (in units of steps; 1 step = $\delta$).

$$\frac{(75(-9)^2 + 87(+9)^2)}{162} = 1.82$$

2b. After 1 step: How many people were -2 step from the starting point? 33

How many people were 0 step from the starting point? 84

How many people were 2 step from the starting point? 45

Calculate the average distance ($x$) from the starting point (in units of steps; 1 step = $\delta$).

$$\frac{(33(-28) + 84(-9) + 45(28))}{162} = 0.188 \approx 0.8$$

Calculate the average ($x^2$) from the starting point (in units of steps; 1 step = $\delta$).

$$\frac{(33(-28)^2 + 84(-9)^2 + 45(28)^2)}{162} = 2.8^2$$

3b. After 1 step: How many people were -3 step from the starting point? 13

How many people were -1 step from the starting point? 65

How many people were +1 step from the starting point? 72

How many people were +3 step from the starting point? 15

Calculate the average distance ($x$) from the starting point (in units of steps; 1 step = $\delta$).

$$\frac{(13(-38) + 65(-18) + 72(18) + 15(38))}{162} = 0.088 \approx 0.8$$

Calculate the average ($x^2$) from the starting point (in units of steps; 1 step = $\delta$).

$$\frac{(13(-38)^2 + 65(-18)^2 + 72(18)^2 + 15(38)^2)}{162} = 2.48^2 \approx 3.8^2$$
Activity 1B: From Random Walk to Diffusion

To build up to an equation for diffusion, we have to generalize the random walk model to \( n \) steps and make a couple definitions.

1. From the previous parts of this activity, do you see a pattern emerging for the values of \( \langle x^2 \rangle \)? Write a general expression for \( \langle x^2 \rangle \) in terms of \( \delta \) and \( n \). Where \( n \) is the number of steps and \( \delta \) is the step size. HINT: the units of \( \langle x^2 \rangle \) should help with the \( \delta \) dependence.

\[
\langle x^2 \rangle = n \delta^2
\]

\[
X_{\text{rms}} = \sqrt{\langle x^2 \rangle} = \sqrt{n} \delta
\]

2. \( \tau \) is the amount of time for a particle to take 1 step. How many steps does a particle take over the amount of time \( t \). That is, express the number of steps \( n \) in terms of \( t \) and \( \tau \).

\[
h = \frac{t}{\tau}
\]

3. Find an expression for \( (x_{\text{rms}})^2 \) in terms of \( t, \tau, \) and \( \delta \). HINT: Replace \( n \) in the expression you came up with in 1. using your result in 2. (recall: \( \langle x^2 \rangle = (x_{\text{rms}})^2 \)).

\[
X_{\text{rms}}^2 = \frac{t}{\tau} \delta^2
\]

\[
X_{\text{rms}} = \delta \sqrt{\frac{t}{\tau}}
\]

The speed of particle's diffusion (or movement from its starting point) increases when the step size \( \delta \) increases and decreases when frequency of steps \( \tau \) increases. So, we can define the Diffusion coefficient \( D \equiv \frac{\delta^2}{2\tau} \). This encapsulates the speed of the diffusion of a particle doing a 1-D random walk. In general, the diffusion coefficient \( D \) is a property of the molecules (or particles) & the fluid.

4. Express \( (x_{\text{rms}})^2 \) in terms of \( D \) and \( t \).

\[
X_{\text{rms}}^2 = \frac{t}{\tau} \frac{\delta^2}{2} \rightarrow 2Dt
\]

\[
X_{\text{rms}} = \sqrt{2Dt}
\]
Activity 2: Diffusion Coefficient – Einstein-Stokes Equation

1. How does the diffusion coefficient $D$ (which describes speed of diffusion) change when you… 
(Hint: it may be helpful to remember for our random walk $D \equiv \frac{\delta^2}{2\tau}$, but now imagine instead of flipping a coin, a new step occurs when the diffusing, spherical object collides with a fluid particle)

a.) Increase $T$, temperature? (circle one) 
$$D \text{ decreases}$$
$$D \text{ increases}$$
$$D \text{ decreases}$$

Einstein-Smoluchowski: $D \propto \frac{k_B T}{f C_{\text{drag}}}$

Now, assume the particles diffusion are spheres

b.) Increase $R$, radius of the particles diffusing? 
$$D \text{ increases}$$
$$D \text{ decreases}$$

c.) Increase $\eta$, viscosity of the fluid? 
$$D \text{ increases}$$
$$D \text{ decreases}$$

$D = \frac{k_B T}{6 \pi \eta R}$

2. Roughly estimate the diffusion constant for sucrose in 300K water ($\eta = 10^{-3}$ Pa·s), assuming that a sucrose molecule is a sphere with radius of 1 nm. Use the Einstein-Stokes equation $D = \frac{k_B T}{6\pi \eta R}$

$k_B = 1.38 \cdot 10^{-23}$ J/K

$$D = \frac{k_B T}{6 \pi \eta R} = \frac{1.38 \times 10^{-23} \cdot 300}{6 \cdot 1 \cdot 10^{-3} \cdot 10^{-9}} \text{ m}^2/\text{s}$$

$$= 2 \times 10^{-10} \text{ m}^2/\text{s}$$

(For comparison, the measured value is $D = 5 \times 10^{-10}$ m$^2$/sec)
Activity 3: Diffusion in 2-D & 3-D and Examples

1. A 2-D random walk consists of two independent random walks along the x and y directions. At each step in the 2-D random walk, the pollen moves randomly in the x and in the y directions.

Fill in the blank for the 2-D random walk expression \( \langle x^2 + y^2 \rangle = \ldots \) \( Dt \)?

(Hint 1: \( \langle x^2 \rangle = 2D t \) in 1-D, Hint 2: x and y are independent)

(a) 1  (b) 3/2  (c) 2  (d) 3  (e) 4

\[ \langle x^2 + y^2 \rangle = \frac{r_{rms}^2}{4} = 4Dt \]

2. A 3-D random walk consists of three independent random walks along the x, y, and z directions. At each step in the 3-D random walk, the pollen moves randomly in the x, y, and z directions.

Fill in the blank for the 3-D random walk expression \( \langle x^2 + y^2 + z^2 \rangle = \ldots \) \( Dt \)?

(Hint: \( \langle x^2 \rangle = 2D t \) in 1-D, Hint 2: x, y, and z are independent)

(a) 1  (b) 2  (c) 3  (d) 4  (e) 6

\[ \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = 3r_{rms}^2 = 6Dt \]

Examples of using the Diffusion equation in 3-D: Sucrose diffusing in water

3. Sucrose has a diffusion constant of about \( D = 5 \times 10^{-10} \) m\(^2\)/sec. If you put a molecule of sucrose in a large container of water without stirring, how long would a molecule take, on average, to diffuse:

(a) \( r_{rms} = 1\) mm?

\[ \frac{r_{rms}^2}{6D} = \frac{(10^{-3} \text{m})^2}{6 \times 5 \times 10^{-10} \text{m}^2/\text{s}} = 3333 \text{s} \]

(b) \( r_{rms} = 1\) cm?

\[ r_{rms} = 10^{-2} \text{m} \]

\[ 6 \text{ min} \]

\[ 1 \text{ hour} \]

**Bonus:** Perfume has a diffusion coefficient in air of \( D = 10^{-6} \) m\(^2\)/s.

Approximately how long will it take for diffusion to spread the odor molecules 10m across the room? Does the answer make sense? Explain the discrepancy.

\[ 200 \text{ days} \]
Diffusion Coefficient – Practice Problems (if time permits)

**Cellular transport:** A protein in cytoplasm has a typical diffusion coefficient of \( D_{\text{protein}} = 10 \mu m^2/sec \).

1. On average, how long does it take a protein to diffuse from one end of a *bacterium* to the other? (assume 1-D random walk) Do you imagine that this a reasonable timescale for transport? Comment please.

\[
\begin{align*}
\text{Yes, reasonable time for transport} & \\
\Rightarrow t &= \frac{x_{\text{rms}}^2}{2D} \\
&= \frac{1 mm^2}{2 \cdot 10 \mu m^2/sec} = \frac{1}{20} \text{ sec}.
\end{align*}
\]

2. On average, how long does it take a protein to diffuse from one side of a large *eukaryotic cell* to the other? (assume 1-D random walk) Do you imagine that this a reasonable timescale for transport? Please comment.

\[
\begin{align*}
\Rightarrow t &= \frac{x_{\text{rms}}^2}{2D} \\
&= \frac{(10 \mu m)^2}{2 \cdot 10 \mu m^2/sec} = 5 \text{ sec}.
\end{align*}
\]

\[\text{No, not reasonable time for cellular transport. Active transport needed in Eukaryotes}\]