* HW4 posted soon, due Wed. Sept 30, collaboration & OH questions allowed.
* Midterm will be posted Mon. Sept 28 after my office hours; due Fri Oct 2.
  No collaboration, no sources except lecture notes + Artin + Axler

  It's meant as a simple check that you know what's going on - it's not meant to be challenging (no * problems) or time-consuming (< 2 hrs for most of you).

  Material: everything up to Lecture 10 (Fri 9/25) ≈ Artin through 4.4 / Axler through ch. 5.

  Do not discuss the midterm or ask about its contents until after end of week, including in office hours, even if you've turned it in.

  E-mail me for clarification requests about the midterm after it gets posted.

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* Quotient spaces: Let V be a vector space over a field k, $U \subset V$ a subspace.

Def: The quotient space $V/U = \{v+U\}$ is the space of cosets of U in V, with addition $(v+U)+(w+U) = (v+w)+U$, scalar multiplication $a(v+U) = av+U$.

The linear map $V \xrightarrow{q} V/U$, $v \mapsto v+U$ is surjective, with kernel $= U$. Hence, we get an exact sequence $0 \to U \to V \to V/U \to 0$.

By the dimension formula ($\dim \ker q + \dim \operatorname{Im} q = \dim V$), we have:

$$\dim(V/U) = \dim V - \dim U.$$

Remarks: 1) given a linear map $V \xrightarrow{\varphi} W$, if $U \subset \ker\varphi$ (ie. $\varphi_{|U} = 0$) then φ factors through V/U, ie.

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "q"'] & & W \\ & V/U \arrow[ur, "\bar\varphi"'] & \end{tikzcd}$$

$\exists \bar\varphi : V/U \to W$ st. $\varphi = \bar\varphi \circ q$.

(define $\bar\varphi(v+U) = \varphi(v)$, this is indep^t of choice of v in coset)

Conversely, given $\bar\varphi \in \operatorname{Hom}(V/U, W)$, $\varphi = \bar\varphi \circ q : V \to W$ has $U \subset \ker\varphi$.

Hence: $\{\varphi \in \operatorname{Hom}(V,W) \mid U \subset \ker\varphi\} \cong \operatorname{Hom}(V/U, W)$.

2) there is a bijection $\{\text{subspaces of } V \text{ containing } U\} \longleftrightarrow \{\text{subspaces of } V/U\}$

$W \subset V, (W \supset U) \longmapsto W/U = \{w+U, w \in W\}$

Conversely, $q^{-1}(\bar W) \subset V \longleftarrow \bar W \subset V/U$

$(U = q^{-1}(0) \subset q^{-1}(\bar W)$ since $0 \in \bar W)$.

<u>Dual spaces</u>: Let V be a vector space over a field k.

<u>Def</u>: The <u>dual vector space</u> is the space of <u>linear functionals</u> on V, ie. linear maps $V \to k$: $V^* = Hom(V,k) = \{\text{linear maps } \ell : V \to k\}$.

<u>Ex</u>: if $V = k^n = \{(x_1, \dots x_n) \mid x_i \in k\}$, any tuple $(a_1, \dots, a_n)$, $a_i \in k$ determines a map $\ell_a : k^n \to k$, $\ell_a(x_1, \dots, x_n) = \sum a_i x_i$.

Conversely, let e_i = standard basis of k^n, given $\ell : k^n \to k$, let $a_i = \ell(e_i)$, then $\ell(x_1, \dots, x_n) = \ell(\sum x_i e_i) = \sum a_i x_i$, ie. $\ell = \ell_a$.

So: $(k^n)^* = \{(a_1 \dots a_n) \mid a_i \in k\} \cong k^n$.

* More generally, given a <u>finite dim'l</u> V and a basis $\{e_1 \dots e_n\}$, then any linear map $\ell : V \to k$ is determined by $\ell(e_i)$, so we get an isomorphism $V^* \cong k^n$, $\ell \mapsto (\ell(e_1), \dots, \ell(e_n))$

  Equivalently, we get a basis of V^* consisting of the linear functionals $e_1^*, \dots, e_n^*$ s.t. $e_i^*(e_i) = 1$ and $e_i^*(e_j) = 0$ for $j \neq i$. (then $\ell = \sum_{i=1}^n \ell(e_i) e_i^*$!).

  This is called the <u>dual basis</u>!

* However, there is <u>not</u> a natural map $V \to V^*$. Despite the above about bases. Each element of the dual basis e_i^* depends not just on e_i, but on <u>all</u> e_j's. There's no such thing as "the dual of a vector".

* On the other hand, we do have a natural map $V \xrightarrow{ev} (V^*)^*$ ("evaluation")
$$v \longmapsto ev_v : V^* \to k, \quad \ell \mapsto \ell(v)$$

* If V is finite-dimensional, then by working in bases $\{e_1 \dots e_n\}$, dual basis $\{e_1^* \dots e_n^*\}$, & double dual basis $\{e_1^{**}, \dots, e_n^{**}\}$, we see that $e_i^{**}(e_j^*) = e_j^*(e_i)$ ✓ and so $ev(e_i) = e_i^{**}$, hence ev is an isom. Thus:

⇒ <u>Prop</u>: If V is finite-dimensional then $V \cong V^{**}$, $v \mapsto (\ell \mapsto \ell(v))$ isomorphism.

* When V is infinite-dimensional, $ev : V \to V^{**}$ is injective, but <u>not</u> an isom!

  The reason is: Assume V has a basis $\{e_i\}_{i \in I}$, so every element of V is uniquely $\sum_{i \in I} x_i e_i$, w/ only finitely many nonzero x_i ($V \simeq \bigoplus_{i \in I} k \cdot e_i$).

Then $\forall (a_i)_{i \in I} \in \prod_{i \in I} k$, $\ell_a : V \to k$, $\sum x_i e_i \mapsto \sum x_i a_i$ is a well defined element of V^*, characterized by $\ell(e_i) = a_i \ \forall i \in I$.

So: $V^* \simeq \prod_{i\in I} k$, which is larger, and the linear functionals e_i^* ($a_i = 1$, $a_j = 0\ \forall j \neq i$) do not span V^*. (Can complete to a basis via Zorn's lemma.) A similar enlargement happens again when passing from V^* to V^{**}.

Def: The <u>annihilator</u> of a subspace $U \subset V$ is $\mathrm{Ann}(U) = \{\ell : V \to k \ / \ \ell_{|U} = 0\} \subset V^*$. (This is a subspace of V^*).

- $\begin{array}{l} V^* \to U^* \\ \ell \mapsto \ell_{|U} \end{array}$ is surjective with kernel $= \mathrm{Ann}(U)$, so $0 \to \mathrm{Ann}(U) \to V^* \to U^* \to 0$, $\ell \mapsto \ell_{|U}$

  This in turn implies $V^*/\mathrm{Ann}(U) \simeq U^*$.
- Also, we've seen above: $\{\ell \in \mathrm{Hom}(V,k)\ /\ U \subset \ker \ell\} \simeq \mathrm{Hom}(V/U, k)$.

  Hence: $\mathrm{Ann}(U) \simeq (V/U)^*$
- Either way, this implies: $\dim \mathrm{Ann}(U) = \dim V - \dim U$.

Def: Given a linear map $\varphi : V \to W$, the <u>transpose</u> of φ, $\varphi^* : W^* \to V^*$ defined as follows: given a linear functional $\ell : W \to k$, composing with $\varphi : V \to W$ gives a linear map $\ell \circ \varphi : V \to k$. Thus, $\begin{array}{l} \varphi^* : W^* \to V^* \\ \ell \mapsto \varphi^*(\ell) := \ell \circ \varphi. \end{array}$

(check: φ^* is linear)

Check:
- given a basis (e_i) of V, elements of V are represented by column vectors X

  $V^* = \mathrm{hom}(V,k)$ —— " —— row vectors Y

  Applying a linear functional $\ell \in V^*$ to a vector $v \in V$ $\Leftrightarrow$ $YX \in k$.
- if $\mathcal{M}(\varphi, (e_i), (f_j)) = A$, then $\mathcal{M}(\varphi^*, (f_j^*), (e_i^*)) = A^T$ transpose matrix

  (This is because: given $\ell \in W^*$ and $v \in V$, $\ell(\varphi(v)) = (\varphi^*(\ell))(v) = YAX$ so φ^*, viewed as operation on row vectors, is $Y \mapsto YA$. Meanwhile the dual bases give a description of elements of V^*, W^* by column vectors, which are the transposes of the row vectors. The claim then follows since $\varphi^*\ell$ as column vector is $(YA)^T = A^T Y^T$.)

Prop: (In the finite dim. case) φ is injective iff φ^* is surjective
φ is surjective iff φ^* is injective

follows from: **Prop:**
(1) $\ker(\varphi^*) = \mathrm{Ann}(\mathrm{Im}\,\varphi)$
(2) $\mathrm{Im}(\varphi^*) = \mathrm{Ann}(\ker \varphi)$ ← assuming finite dim.

Proof: (1) $\ell \in \mathrm{Ann}(\mathrm{Im}\,\varphi) \iff \ell(\varphi(v))=0\ \forall v\in V \iff \varphi^*(\ell)=\ell\circ\varphi=0 \iff \ell\in\mathrm{Ker}\,\varphi^*$.

(2) If $\ell'=\varphi^*(\ell)\in\mathrm{Im}(\varphi^*)$ then $\ell'=\ell\circ\varphi$ so $\ell'|_{\mathrm{Ker}\,\varphi}=0$. So $\mathrm{Im}(\varphi^*)\subset\mathrm{Ann}\,\mathrm{Ker}\,\varphi$.

Dim. formula and (1) imply $\mathrm{rank}(\varphi^*)=\mathrm{rank}(\varphi)$, hence the inclusion is an equality. □

Linear operators:

A linear operator on V (aka endomorphism of V) is a linear map $\varphi: V\to V$.

Notation: $\mathrm{End}(V)=\mathrm{Hom}(V,V)$.

* When using a basis to express $\varphi\in\mathrm{Hom}(V,V)$ by a (square) matrix, we want to use the same basis on each side: $A=\mathcal{M}(\varphi,(e_i),(e_i))$ transforms by $P^{-1}AP$.
* New thing: we can compose linear operators with each other $\varphi\psi=\varphi\circ\psi: V\to V$ or with themselves, $\varphi^n=\underbrace{\varphi\circ\dots\circ\varphi}_{n\text{ times}}$, or even apply polynomials:
$$p=\sum a_n x^n \rightsquigarrow p(\varphi)=\sum a_n\varphi^n: V\to V.$$

$\mathrm{Hom}(V,V)$ is a (noncommutative) ring.

* given vector spaces V_1,V_2 and linear operators $\varphi_i: V_i\to V_i$, we can define
$$\varphi=\varphi_1\oplus\varphi_2: V_1\oplus V_2\to V_1\oplus V_2 \quad \text{operator on } V=V_1\oplus V_2.$$
The operator φ leaves the subspaces $V_1, V_2\subset V$ invariant: $\varphi(V_i)\subset V_i$, and working in a basis of V st. $e_1\dots e_m\in V_1$, $e_{m+1}\dots e_n\in V_2$, the matrix of φ is block diagonal: $\left(\begin{array}{c|c}\varphi_1 & 0\\ \hline 0 & \varphi_2\end{array}\right)$. Conversely, if $V=V_1\oplus V_2$ and $\varphi(V_i)\subset V_i$ then φ is of this form.

More generally, if we only assume $\varphi: V\to V$ and $V_1\subset V$ is invariant ($\varphi(V_1)\subset V_1$) but not necess. V_2, then the matrix of φ would be block triangular: $\left(\begin{array}{c|c}\varphi|_{V_1} & *\\ \hline 0 & *\end{array}\right)$

So: a typical way to study $\varphi: V\to V$ is to look for invariant subspaces.

* If $U\subset V$ is invariant and $\dim U=1$ (so: $U=k.v$ for some $v\in V$), then necessarily $\varphi(v)=\lambda v$ for some $\lambda\in k$.
In this case v is called an eigenvector of φ, and λ is called the eigenvalue corresponding to v.

- If we can find a basis of V consisting of eigenvectors of φ, then we have <u>diagonalized</u> φ, i.e. found a basis where its matrix is <u>diagonal</u>

$$\varphi(v_i) = \lambda_i v_i \qquad \begin{array}{c} \quad v_1 \ \cdots \ v_n \\ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \end{array}$$

This is the best outcome, but not always possible!

<u>Ex</u>: $V = \mathbb{R}^2$, $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ has eigenvectors $(1,0)$, $(0,1)$ (or any multiples) with eigenvalues λ, μ.

However $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only one eigenvector $(1,0)$ (up to scaling!) with eigenvalue 1, <u>NOT</u> diagonalizable.

Next time, we'll learn more about eigenvectors, invariant subspaces, etc.