Last time, we talked about linear operators $\varphi: V \to V$, their invariant subspaces ($U \subset V$ st. $\varphi(U) \subset U$), and eigenvectors ($v \neq 0$ st. $\varphi(v) = \lambda v$, ie. $v \in \mathrm{Ker}(\varphi - \lambda I)$).

Over any field:

- eigenvectors need not exist; eigenvectors for distinct λ are linearly independent;
- if $\exists$ $n = \dim V$ distinct eigenvalues then φ is diagonalizable: $\exists$ basis st. $\mathcal{M}(\varphi) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

We saw that, over alg. closed fields, eg. $\mathbb{C}$:

- every operator has at least one eigenvector.
- $\exists$ basis st. $\mathcal{M}(\varphi)$ is upper triangular $\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$
  ($\Leftrightarrow$ the subspaces $V_i = \mathrm{span}(v_1, \dots, v_i)$ are all invariant).
- $\varphi - \lambda I$ is invertible $\Leftrightarrow \lambda \notin \{\lambda_1, \dots \lambda_n\}$, so the diagonal entries are the eigenvalues of φ!

<u>Today's goal</u>: further study of invariant subspaces & eigenvalues for linear operators over alg. closed k, especially $\mathbb{C}$ – <u>Jordan normal form</u>.

(This is Axler ch. 8 – we'll return to the skipped chapters 6 & 7 soon).

Recall $\ker(\varphi) = \{v \in V / \varphi(v) = 0\}$.

<u>Def</u>: the <u>generalized kernel</u> of φ is $\mathrm{gker}(\varphi) = \{v \in V / \exists m > 0 \text{ st. } \varphi^m(v) = 0\}$

These are all the vectors that are eventually sent to 0 by repeatedly aplying φ.

<u>Observe</u>: $0 \subset \ker \varphi \subset \ker(\varphi^2) \subset \dots$ (since: $\varphi^m(v) = 0 \Rightarrow \varphi^{m+1}(v) = 0 \dots$)
if $\ker(\varphi^m) = \ker(\varphi^{m+1})$ then the sequence remains constant after that!

($\underline{\text{Pf}}$: $\ker \varphi^{m+1} = \varphi^{-1}(\ker \varphi^m)$ so $\ker \varphi^m = \ker \varphi^{m+1} \Rightarrow \ker \varphi^{m+1} = \varphi^{-1} \ker \varphi^m = \varphi^{-1} \ker \varphi^{m+1} = \ker \varphi^{m+2}$)

Since the sequence stops increasing after at most $n = \dim V$ steps, $\mathrm{gker}(\varphi) = \ker \varphi^n$.

<u>Example</u>: $\varphi: k^2 \to k^2$, $e_1 \mapsto 0$, $e_2 \mapsto e_1$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ Then $\ker(\varphi) = k \cdot e_1$, but $\ker(\varphi^2) = \mathrm{gker}(\varphi) = k^2$.

<u>Lemma</u>: if $\mathrm{gker}(\varphi) = \ker(\varphi^m)$ then $V = \ker(\varphi^m) \oplus \mathrm{Im}(\varphi^m)$.

<u>Pf</u>: If $v = \varphi^m(u) \in \mathrm{Im}(\varphi^m) \cap \ker(\varphi^m)$ then $\varphi^m(v) = \varphi^{2m}(u) = 0 \Rightarrow u \in \ker \varphi^{2m} \overset{\text{using } \ker \varphi^m = \mathrm{gker}.}{=} \ker \varphi^m$, so $v = \varphi^m(u) = 0$. Hence $\mathrm{Im}(\varphi^m) \cap \ker(\varphi^m) = \{0\}$. By dimension formula, $\mathrm{Im} \oplus \ker = V$. □

<u>Def</u>: Say φ is <u>nilpotent</u> if $\exists m > 0$ st. $\varphi^m = 0$, ie. $\mathrm{gker}(\varphi) = V$.

* Now we can do the same thing to eigenspaces:

Def: $v \in V$ is a <u>generalized eigenvector</u> with generalized eigenvalue λ if $v \in \mathrm{gker}(\varphi - \lambda I)$ ie. $\exists\, m > 0$ st. $(\varphi - \lambda I)^m v = 0$. Call $\mathrm{gker}(\varphi - \lambda I)$ the <u>generalized eigenspace</u>

Def: The <u>multiplicity</u> of the eigenvalue λ is the dimension of the generalized eigenspace $V_\lambda = \mathrm{gker}(\varphi - \lambda I)$. $\left(= \ker(\varphi - \lambda I)^n\right)$.

In a basis where the matrix of φ is triangular, this is the number of times λ appears on the diagonal! (This will be clearer later...)

Prop.1: $V_\lambda = \ker(\varphi - \lambda I)^n$ and $W_\lambda = \mathrm{Im}(\varphi - \lambda I)^n$ are invariant subspaces of φ, and $V = V_\lambda \oplus W_\lambda$.

Proof:
- let $v \in V_\lambda$, then $(\varphi - \lambda I)^n v = 0$, hence $\varphi(\varphi - \lambda I)^n v = 0$. But $\varphi - \lambda I$ commutes with φ, so this implies $(\varphi - \lambda I)^n \varphi v = 0$, hence $\varphi(v) \in V_\lambda$.
- if $v = (\varphi - \lambda I)^n u \in W_\lambda$ then $\varphi(v) = \varphi(\varphi - \lambda I)^n u = (\varphi - \lambda I)^n \varphi(u) \in \mathrm{Im}(\varphi - \lambda I)^n = W_\lambda$.
- the lemma above, applied to $\varphi - \lambda I$, says $V = \ker(\varphi - \lambda I)^n \oplus \mathrm{Im}(\varphi - \lambda I)^n$. $\square$

Prop2: The subspaces $V_\lambda \subset V$ are independent: $\sum v_i = 0$, $v_i \in V_{\lambda_i} \Rightarrow v_i = 0\ \forall i$.

Proof: Assume $\sum_{i=1}^{\ell} v_i = 0$, $v_i \in V_{\lambda_i}$, λ_i distinct. We'll show $v_1 = 0$ (same for the others).
If $v_1 \neq 0$, let $k \geq 0$ be the largest integer st. $(\varphi - \lambda_1 I)^k v_1 = w \neq 0$
(but $(\varphi - \lambda_1 I)^{k+1} v = 0$, so $\varphi(w) = \lambda_1 w$).
Observe: $(\varphi - \lambda_\ell I)^n \cdots (\varphi - \lambda_2 I)^n (\varphi - \lambda_1 I)^k \big(\overbrace{v_1 + \cdots + v_\ell}^{=0}\big) = 0$

is the sum of $(\varphi - \lambda_\ell I)^n \cdots (\varphi - \lambda_2 I)^n w = \prod_{j=2}^{\ell} (\lambda_1 - \lambda_j)^n w \neq 0$

and $(\varphi - \lambda_\ell I)^n \cdots (\varphi - \lambda_2 I)^n (\varphi - \lambda_1 I)^k v_j = 0 \quad \forall j \geq 2$

(because the operators $(\varphi - \lambda I)$ commute, and $(\varphi - \lambda_j I)^n v_j = 0$).

Contradiction, hence $v_1 = 0$, and similarly $v_i = 0\ \forall i$. $\square$

Thm.: If k is alg. closed, V finite-dim. vect space over k, $\varphi : V \to V$, then V decomposes into the direct sum of the generalized eigenspaces V_λ of φ, $V = \bigoplus V_\lambda$.

Proof: By induction on $\dim V$! (the result is clear for $\dim V = 1$). Assume the result holds up to dimension $n-1$, and consider the case $\dim V = n$.
We've seen before: k alg. closed $\Rightarrow$ φ has at least one eigenvalue λ_1
Let $V_{\lambda_1} = \mathrm{gker}(\varphi - \lambda_1 I) = \ker((\varphi - \lambda_1 I)^n)$, $U = W_{\lambda_1} = \mathrm{Im}(\varphi - \lambda_1 I)^n$.

By prop. 1 above, V_{λ_1} and U are invariant subspaces, and $V = V_{\lambda_1} \oplus U$.

Since $\dim U < \dim V$, induction $\Rightarrow$ U decomposes into generalized eigenspaces for $\varphi_{|U}$,
$U = U_{\lambda_2} \oplus \dots \oplus U_{\lambda_\ell}$, $\lambda_2 \dots \lambda_\ell$ eigenvalues of $\varphi_{|U}$ ($\Leftrightarrow$ eigenvalues of φ with an eigenvector $\in U$)

$$U_{\lambda_j} = \ker(\varphi_{|U} - \lambda_j I)^n = \ker(\varphi - \lambda_j I)^n \cap U = V_{\lambda_j} \cap U$$

Moreover, $\varphi_{|U}$ doesn't have λ as eigenvalue (since $\ker(\varphi - \lambda I)^n \cap U = 0$), so $\lambda \notin \{\lambda_2 \dots \lambda_\ell\}$.

Now: $U_{\lambda_j} \subset \ker(\varphi - \lambda_j I)^n = V_{\lambda_j}$, and $V = V_{\lambda_1} \oplus U = V_{\lambda_1} \oplus U_{\lambda_2} \oplus \dots \oplus U_{\lambda_\ell}$.
Since the gen^d eigenspaces V_{λ_j} contain U_{λ_j} $\forall j \geq 2$, we find that $V_{\lambda_1}, \dots V_{\lambda_\ell}$ span V;
and they are independent by Prop. 2, hence $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_\ell}$.
(and in fact $V_{\lambda_j} = U_{\lambda_j}$ $\forall j \geq 2$; in other terms, $\operatorname{Im}(\varphi - \lambda_i I)^n = \bigoplus_{j \neq i} \ker(\varphi - \lambda_j I)^n$.)

□

* The decomposition $V = \oplus V_{\lambda_i}$ gives us bases in which φ is given by a block diagonal matrix
$$\begin{pmatrix} \varphi_{|V_{\lambda_1}} & & 0 \\ & \varphi_{|V_{\lambda_2}} & \\ 0 & & \ddots \ \varphi_{|V_{\lambda_\ell}} \end{pmatrix}$$
* Moreover, $\varphi_{|V_{\lambda_i}}$ can be represented by a triangular matrix in a suitable basis for V_{λ_i} (using thm seen last time), and since its only eigenvalue is λ_i, the diagonal entries are all λ_i! So: $\varphi \sim$
$$\begin{pmatrix} \begin{matrix}\lambda_1 & * \\ 0 & \lambda_1\end{matrix} & & 0 \\ & \begin{matrix}\lambda_2 & * \\ 0 & \lambda_2\end{matrix} & \\ 0 & & \ddots \ \begin{matrix}\lambda_\ell & * \\ 0 & \lambda_\ell\end{matrix} \end{pmatrix}$$
* We can do more with the blocks $\begin{pmatrix} \lambda_i & * \\ 0 & \lambda_i \end{pmatrix}$ but this requires further study of nilpotent operators (note: $\varphi_{|V_{\lambda_i}} - \lambda_i I$ nilpotent!)

<u>Nilpotent operators</u>: let $\varphi : V \to V$ nilpotent (ie. $\varphi^m = 0$ for some $m \leq \dim V$).
(This part works over any field)

<u>Goal</u>: find a "nice" basis of V for φ.

<u>Observe</u>: if $\dim V = 2$, there are 2 cases: either $\varphi = 0$; or $\varphi^2 = 0$ but $\varphi \neq 0$.
In second case: let $v \notin \ker \varphi$, then $\varphi(v) = u \in \ker \varphi$ so u, v are independent and form a basis, in which $\mathcal{M}(\varphi) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Jordan's method generalizes this to higher dimensions:

<u>Prop</u>: $\exists$ basis of V: $\{\varphi^{m_1}(v_1), \varphi^{m_1 - 1}(v_1), \dots, v_1, \dots, \varphi^{m_k}(v_k), \dots, v_k\}$ where $\varphi^{m_i + 1}(v_i) = 0$ $\forall i$

in which $\mathcal{M}(\varphi) = \begin{pmatrix} \begin{matrix}0 & 1 & 0 \\ & \ddots & 1 \\ 0 & & 0\end{matrix} & & 0 \\ & \begin{matrix}0 & 1 & \\ & \ddots & 1 \\ & & 0\end{matrix} & \\ 0 & & \ddots \ \begin{matrix}0 & 1 & \\ & \ddots & 1 \\ & & 0\end{matrix} \end{pmatrix}$

block diagonal built from <u>nilpotent Jordan blocks</u> $\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$
(each basis element $\mapsto$ previous one)
first basis elt $\mapsto 0$

Proof: Recall $0 \subset \ker\varphi \subset \ker\varphi^2 \subset \ldots \subset \ker\varphi^m = V$ ↳ assume this is the smallest m, ie. $\varphi^m = 0$ but $\varphi^{m-1} \neq 0$.

Claim: if a subspace $U \subset \ker(\varphi^{k+1})$ satisfies $\ker(\varphi^k) \cap U = \{0\}$ ($k \geq 1$), then $\varphi_{|U}$ is injective, $\varphi(U) \subset \ker(\varphi^k)$, and $\ker(\varphi^{k-1}) \cap \varphi(U) = \{0\}$.

Indeed: $\forall\, v \in U,\ v \neq 0 \Rightarrow \begin{cases} \varphi^k(v) \neq 0. \\ \varphi^{k+1}(v) = 0 \end{cases}$ In particular $\varphi(v) \neq 0$, ie. $\ker(\varphi_{|U}) = \{0\}$, injective.
Also, $\varphi^k(\varphi(v)) = 0 \Rightarrow \varphi(v) \in \ker\varphi^k$
and $\varphi^{k-1}(\varphi(v)) = \varphi^k(v) \neq 0 \Rightarrow \varphi(v) \notin \ker\varphi^{k-1}$.

First step: let U_m st. $\ker(\varphi^m) = V = \ker(\varphi^{m-1}) \oplus U_m$
& pick a basis $(v_{m,1}, \ldots, v_{m,k_m})$ of U_m [these will yield Jordan blocks of size m!]

(eg: start from a basis of $\ker\varphi^{m-1}$, extend to basis of V by adding vectors $v_{m,1}, \ldots, v_{m,k_m}$, and let U_m be their span).

Now by the claim, $v_{m-1,1} = \varphi(v_{m,1}), \ldots, v_{m-1,k_m} = \varphi(v_{m,k_m})$ are linearly independent, and their span is $\subset \ker(\varphi^{m-1})$ but independent of $\ker(\varphi^{m-2})$.
Start from a basis of $\ker(\varphi^{m-2})$, add $v_{m-1,1}, \ldots, v_{m-1,k_m}$ and complete to a basis of $\ker(\varphi^{m-1})$ by adding some other vectors $v_{m-1,k_m+1}, \ldots, v_{m-1,k_{m-1}}$ (if needed: could have $k_{m-1} = k_m$). (these will yield blocks of size $m-1$).
Let $U_{m-1} = \operatorname{span}(v_{m-1,1}, \ldots, v_{m-1,k_{m-1}})$. Then $\ker(\varphi^{m-1}) = \ker(\varphi^{m-2}) \oplus U_{m-1}$.
And so on: given $U_j = \operatorname{span}(v_{j,1} \ldots v_{j,k_j})$ with $\ker\varphi^j = \ker\varphi^{j-1} \oplus U_j$, take $v_{j-1,i} = \varphi(v_{j,i})$ for $1 \leq i \leq k_j$ and extend by adding vectors as needed to build U_{j-1}. This eventually gives a basis of $V = U_1 \oplus \ldots \oplus U_m$, and rearranging it as $(v_{1,1}, \ldots, v_{m,1}, v_{1,2}, \ldots)$ we get the result. □

We now combine our results to arrive at the

Jordan normal form: V finite dim. vector space over k alg. closed (→ eg. $\mathbb{C}$), $\varphi \in \operatorname{Hom}(V,V)$
$\Rightarrow \exists$ basis of V in which the matrix of φ is block-diagonal, with each block a Jordan block $\begin{pmatrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{pmatrix}$.

Rmks: • size 1 Jordan block: (λ), size 2: $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, … φ is diagonalizable $\Leftrightarrow$ all the blocks have size 1.
• the values of λ that appear are exactly the eigenvalues of φ. There may be several blocks with the same λ; their direct sum is the generalized eigenspace V_λ.
• proof: we've seen $V = \oplus V_\lambda$ generalized eigenspaces; now $\varphi_{|V_\lambda} - \lambda I$ is nilpotent, so can decomposed into nilpotent Jordan blocks $\varphi_{|V_\lambda} - \lambda I = \oplus \begin{pmatrix} 0 & 1 & \\ & \ddots & 1 \\ & & 0 \end{pmatrix}$, so $\varphi_{|V_\lambda} = \oplus \begin{pmatrix} \lambda & 1 & \\ & \ddots & 1 \\ & & \lambda \end{pmatrix}$.

Next time:
- → characteristic polynomial & minimal polynomial
- → real operators?
- → digression: categories & functors
- → start: bilinear forms & inner product spaces.