Recitation 1

Thursday September 9, 2022

Welcome to 18.061. In this recitation, we will review some key points from Lecture 1 and do a few practice problems together. The remaining exercises are left to you for reference. Solutions will be released before next recitation.

A friendly reminder to 1) sign up for Piazza which is our primary Q&A platform, and 2) install and try out Julia at your convenience. Do not worry if you’re new to the language. We will have a Julia session at some point.

1 Recap

We went over some basic definitions and concepts of linear algebra.

1.1 Vectors

• A vector is a tuple of numbers.
• The dimension of the vector is the size of the tuple.
• A vector has a magnitude and a direction.
• The magnitude of a $n$-dimensional real vector $\mathbf{u}$, or $\|\mathbf{u}\|$, is the scalar $\sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$, where each $u_i$ is an entry of the vector $\mathbf{u}$.
• Unit vectors have a magnitude of 1.

1.2 Vector/Scalar Operations

• We can multiply a vector by a scalar (scalar multiplication).

$$\lambda \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \end{bmatrix}$$

• We can add two vectors of the same dimension (vector addition).

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$$

• Another operation called the inner product takes two vectors of the same dimension and gives a scalar. In particular, given two $n$-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$, their inner product, denoted by $\mathbf{u} \cdot \mathbf{v}$, is

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$
1.3 Matrices

We can also represent data (or operations on data) by using a matrix, which is a rectangular array of numbers. Vectors and matrices are closely related. If the dimensions are compatible, we can take a matrix-vector product:

\[
\begin{bmatrix}
a & b \\
c & d \\
e & f
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
ax + by \\
cx + dy \\
ex + fy
\end{bmatrix}
\]

(4)

The result can also be interpreted as a linear combination of the columns of the matrix:

\[
x \begin{bmatrix}
a \\
c \\
e
\end{bmatrix}
+ y \begin{bmatrix}
b \\
d \\
f
\end{bmatrix}
\]

(5)
2 Exercises

Time: Problem 1: 5 minutes, Problem 2-4: 5 minutes, Problem 5: 5 minutes.

2.1 During recitation

1. Consider the expression

\[ a \begin{bmatrix} 1 \\ 3 \\ x \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} \]

(a) Find \( \begin{bmatrix} m \\ n \end{bmatrix} \) when \( a = 2, b = 4 \).
(b) Find the inner product between the vectors \( x \) and \( y \).
(c) Re-write the equation above in matrix-vector product form.
(d) Find values for \( a \) and \( b \), such that \( m = 5, n = 10 \).

*Hint: write it as a system of linear equations.*

2. Let

\[ A = \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 & 7 \\ -2 & -5 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

Which of the following matrix products are defined?
(a) \( AB \)
(b) \( CA \)
(c) \( ACB \)
(d) \( \ldots \)

3. True or False: A vector (with real coordinates) has magnitude 0 if and only if it is the zero-vector, i.e. it consists of only zeroes.

4. True or False: If a vector is multiplied by a scalar \( c \), its magnitude also changes by a factor of \( |c| \).

5. In this exercise, we compute matrices associated to certain geometric transformations of vectors:
(a) Find a \( 2 \times 2 \) matrix such that when you multiply a 2-dimensional vector by it, the result is the reflection of the vector across the origin.
(b) Find a \( 3 \times 3 \) matrix such that when you multiply a 3-dimensional vector by it, it swaps the second and third coordinates of the vector.
2.2 After recitation (optional)

6. Let
\[ A = \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 & 7 \\ -2 & -5 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}. \]

Which of the following matrix/vector products are well defined?

(a) BA
(b) AC
(c) BC
(d) ABADDAD
(e) AAA
(f) ABC
(g) ABD
(h) BACD
(i) ACBD
(j) BACBACBACBACBAC

7. True or False: For any vector \( u \), we have \( \|u\|^2 = u \cdot u \).

8. True or False: Let \( a \) and \( b \) be vectors of the same dimension. Then \( a \cdot b = b \cdot a \) if and only if \( a = b \).

9. Consider the following matrix \( A \) and vector \( y \). Does there exist a vector \( x \) such that \( Ax = y \)?

\[ A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \]

10. If you have a 4 \( \times \) 4 matrix \( A \), which 4-dimensional vector \( x \) can you choose such that \( Ax \) is the second column of \( A \)?

11. If you have a 4-dimensional vector \( x \), which 4\( \times \)4 matrix \( A \) can you choose such that \( Ax \) has its first/second/third/fourth entry being identical/double/triple/quadruple of \( x \)'s.

12. Install Julia. Test it out by defining a few vectors and matrices, and trying to add/multiply them. Also try to add/multiply things of the wrong dimensions, and see what happens.

For instance, you can define a matrix \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) and a vector \( x = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \), and multiply them by typing the following:

\[ A = [1 \ 2 \ ; \ 3 \ 4] \]
\[ x = [5 \ ; \ 6] \]
\[ A*x \]

More information on how to download and try Julia online can be found in Piazza.
3 Solutions

1. (a) \[ \begin{bmatrix} m \\ n \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 + (-4) \\ 6 + 8 \end{bmatrix} = \begin{bmatrix} -2 \\ 14 \end{bmatrix} \]

(b) \[ x \cdot y = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 1 \cdot (-1) + 2 \cdot 3 = 5. \]

(c) \[ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}. \]

(d) \[ \begin{bmatrix} 5 \\ 10 \\ 3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \end{bmatrix} = a \begin{bmatrix} 0 \\ 3a \end{bmatrix} + b \begin{bmatrix} -3 \\ 2b \end{bmatrix} = \begin{bmatrix} a - b \\ 3a + 2b \end{bmatrix}. \] Therefore, we have \[ a - b = 5 \text{ and } 3a + 2b = 10. \] These equations together solve to \[ a = 4 \text{ and } b = -1. \]

2. Matrices/vectors products are defined if and only if inner dimensions match.

Defined: (a) AB, (c) ACB, (d) CCCCCCCCCCCCCC.

Undefined: (b) CA.

3. True. If a vector \( v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \) has magnitude 0, then \( v_1^2 + v_2^2 + \cdots + v_n^2 = 0 \) which occurs if and only if \( v_1 = v_2 = \cdots = v_n = 0. \)

4. True. Let’s say we have a vector \( v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \). Then \( cv = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix} \). So we have \[
\|cv\| = \sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2} = \sqrt{c^2(v_1^2 + v_2^2 + \cdots + v_n^2)} = |c| \cdot \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = |c| \cdot \|v\|. \]

5. (a) A 2-dimensional vector \( \begin{bmatrix} x \\ y \end{bmatrix} \) has a reflection across the origin \[
\begin{bmatrix} -x \\ -y \end{bmatrix} = x \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \]

Therefore, the matrix representing this transformation is \( \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \)

(b) A 3-dimensional vector \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) must yield a result \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) which is \[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \]
Therefore, the matrix representing this transformation is 
\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \]

6. Matrices/vectors products are defined if and only if inner dimensions match.

Defined: (a) BA, (b) AC, (g) ABD, (i) ACBD, (j) BACBACBACBACBAC.

Undefined: (c) BC, (d) ABADDAD, (e) AAA, (f) ABC, (h) BACD.

7. True. Both terms are identical since
\[
\|u\|^2 = \left( \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \right)^2 = u_1^2 + u_2^2 + \cdots + u_n^2
\]

8. False. The values of \( a \cdot b = \sum_{i=1}^{n} a_i b_i \) and \( b \cdot a = \sum_{i=1}^{n} b_i a_i \) are the same regardless of \( a_i \)'s and \( b_i \)'s.

9. No, there are no possible solutions because any vector multiplied by the last row of \( A \) will always equal 0.

10. Let’s say \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \). Then

\[
Ax = x_1 \cdot (A's \ 1^{st} \ column) + x_2 \cdot (A's \ 2^{nd} \ column) + \\
x_3 \cdot (A's \ 3^{rd} \ column) + x_4 \cdot (A's \ 4^{th} \ column).
\]

Since we want \( Ax \) to be just the second column of \( A \), it follows that \( x_1 = x_3 = x_4 = 0 \) and \( x_2 = 1 \). Therefore, \( x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \).

11. Let’s say \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \). Then

\[
Ax = x_1 \cdot (A's \ 1^{st} \ column) + x_2 \cdot (A's \ 2^{nd} \ column) + \\
x_3 \cdot (A's \ 3^{rd} \ column) + x_4 \cdot (A's \ 4^{th} \ column).
\]

On the other hand, we want
\[
Ax = \begin{bmatrix} x_1 \\ 2x_2 \\ 3x_3 \\ 4x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}.
\]
Therefore, the first/second/third/fourth columns are 
\[
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}
\]
respectively. This means 
\[
A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}
\]