Recitation 15

Tuesday Nov 1, 2022

1 TLDR

1.1 Eigenvalues and Eigenvectors

1.1.1 Definition

Let $A$ be a square $n \times n$ matrix. A vector $x \in \mathbb{R}^n$ is called an eigenvector iff $Ax = \lambda x$ for some scalar $\lambda$. All the scalars $\lambda$'s satisfying this equation is called an eigenvalue.

1.1.2 Characteristic polynomial

Given a square matrix $A \in \mathbb{R}^{n \times n}$. Its characteristic polynomial $p(\cdot)$ is defined as

$$p(\lambda) = \det(\lambda I - A),$$

which is a degree-$n$ polynomial in $\lambda$. $\lambda$ is an eigenvalue of $A$ if and only if it is a root of $p(\lambda)$ – that is $p(\lambda) = 0$.

A matrix $A \in \mathbb{R}^{n \times n}$ can have up to $n$ distinct eigenvalues – as they are roots of a degree-$n$ polynomial $p(\lambda) = 0$.

1.1.3 Properties of Eigenvalues and Eigenvectors

1. The determinant of $A$ is equal to the product of the eigenvalues of $A$.

2. The trace of $A$ (sum of diagonal elements) is equal to the sum of the eigenvalues of $A$.

3. If $v_1, \ldots, v_k$ are eigenvectors associated to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $v_1, \ldots, v_k$ are linearly independent.
2 Exercises

1. T/F
   (a) If $A$ has eigenvalue 0, then $A$ is singular.
   (b) If $v$ is an eigenvector of $A$, then $cv$ where $c$ is a scalar, is also a eigenvector of $A$.
   (c) If $\lambda$ is an eigenvalue of $A$, then $\lambda^2$ is an eigenvalue of $A^2$.
   (d) If $(v_1, v_2, v_3)$ is an eigenvector of $A$, then $(v_1^2, v_2^2, v_3^2)$ is an eigenvector of $A^2$.
   (e) If $\lambda$ is an eigenvalue of $A$, then $\lambda$ is also an eigenvalue of $A^T$.
   (f) If we add 1 to every entry of $A$, the eigenvalues of $A$ will all increase by 1.
   (g) If we shift $A$ by $I$, the eigenvalues of $A$ will all shift by 1.
   (h) The real eigenvalues of $A^T A$ must be non-negative.
   (i) If two rows in matrix $A$ are switched, the eigenvalues remain the same.
   (j) If every row of $A$ sum up to $k$, then $k$ is an eigenvalue of $A$.
   (k) If every column of $A$ sum up to $k$, then $k$ is an eigenvalue of $A$.

2. 
   \[ A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \]
   (a) Find the eigenvalues and eigenvectors of $A$.
   (b) Find the eigenvalues and eigenvectors $2A$.
   (c) Find the eigenvalues and eigenvectors $A^2$.
   (d) Find the eigenvalues and eigenvectors $A^{-1}$.
   (e) Find the eigenvalues and eigenvectors $A + 4I$.

3. 
   \[ A = \begin{bmatrix} 4 & 1 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{bmatrix} \]
   (a) Write the characteristic polynomial for $A$, and find the eigenvalues.
   (b) Find the eigenvector corresponding to each eigenvalue.

4. 
   \[ A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \]
   (a) Find the eigenvalues and eigenvectors of $A$. What would you predict to be the eigenvalues of $A^\infty$?
   (b) $A^2 = \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix}$ Find the eigenvalues and eigenvectors of $A^2$ using answers from part (a).
   (c) $A^\infty = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$ Find the eigenvalues and eigenvectors of $A^\infty$. Does this match your prediction from part (a)?