Def. A perfect matching is a matching s.t. every vertex appears in exactly one edge.

(matchings/ perfect matchings also apply to graphs that are not bipartite.)

What conditions guarantee a perfect matching exists?

G = (V, E) bipartite \[ V = L \cup R \]
L = set of vertices on left side (allow matching to \[ R = \] set on right side be perfect if all vertices in \[ L \] are matched)

\[ N(S) = \{ v \mid (u, v) \in E, u \in S \}, \text{ set of neighbors of } S \]

The matching condition - If \( S \subseteq L \), \( |N(S)| \geq |S| \)

Theorem. A bipartite graph has a perfect matching if and only if it satisfies the matching condition.

Proof. If a perfect matching exists, let \( S \) be any \( d \in L \). Let \( M(S) \) be set of vertices matched with \( S \). \( |M(S)| = |S| \), \( M(S) \subseteq N(S) \), \( |S| = |M(S)| \leq |N(S)| \), and matching condition holds.

Other direction: induction on \( |L| \)
Base case, \( |L| = 1 \), match \( v \in L \) with anything in \( N(v) \).

Inductive step - Assume statement holds if \( |L| \leq k \).

If \( |L| = k+1 \)
Either \( |N(S)| = |S| \) for some \( S \subseteq L \) - Case 2
or \( |N(S)| > |S| \) for every \( S \subseteq L \) - Case 1

For every \( S \subseteq L \) let \( L = N(S) \).
Then \( G = (V \cup \{u, v\}, E \text{ without edges } u, v) \) contains a perfect matching.

Use the matching from the inductive step.
Case 2: $|N(S)| \leq |S|$ for some $S$ \\
with $S \notin N(S)$ \\
$L \setminus S \subseteq R \setminus N(S)$ \\
both by inductive hypothesis

Assume you can't match $L \setminus S \subseteq R \setminus N(S)$. \\
By inductive hypothesis, there exist $S_0 \subseteq L \setminus S$ s.t. \\
$\{ v \mid \{v, w\} \in E, w \in S_0, w \in R \setminus N(S) \} = N_0(S_0)$ \\
set of neighbors of $S_0$ in $R \setminus N(S)$ \\

$|N_0(S_0)| < |S_0|$ \\

Then, $N(S \cup S_0) = N_0(S_0) \cup N(S)$ \\
$|N(S \cup S_0)| = |N_0(S_0)| + |N(S)| < |S_0| + |S| = |S \cup S_0|$ 
contradicts matching \\

disjoint sum rule \\
same rule \\
condition in the original graph